Hopf algebra $\mathcal{K}_n$ and universal Chern classes

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Abstract

We construct a variant $\mathcal{K}_n$ of the Hopf algebra $\mathcal{H}_n$, which acts directly on the noncommutative model for the generic space of leaves rather than on its frame bundle. We prove that the Hopf cyclic cohomology of $\mathcal{K}_n$ is isomorphic to that of the pair $(\mathcal{H}_n, \mathrm{GL}_n)$ and thus consists of the universal Hopf cyclic classes. We then realize these classes in terms of geometric cocycles.

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Introduction

The application of Connes' cyclic *Ext* functor [4] to the cohomology of Hopf algebras, originally employed to compute the local index formula [6] for hypoelliptic operators on spaces of leaves of foliations [7], has stimulated the interest in developing a theory of Hopf cyclic characteristic classes in the framework of noncommutative geometry. To this end the geometric characteristic classes of foliations (see e.g. [3]) have been gradually reconfigured in the context of Hopf cyclic cohomology [16, 17, 22, 19, 20], which holds the potential of being applicable to other noncommutative spaces (e.g. [10]).

In this paper we construct a variant $\mathcal{K}_n$ of the Hopf algebra $\mathcal{H}_n$, which acts directly on the noncommutative model for the generic space of leaves rather than on its frame bundle. Supplementing our earlier techniques with those in [21], we prove that the (absolute) Hopf cyclic cohomology of $\mathcal{K}_n$ is also the repository of the universal Hopf cyclic classes. By a construction parallel to that in [19] we then realize these classes in terms of concrete geometric cocycles, in the spirit of the Chern-Weil theory.

The paper is organized as follows. In §1 we introduce the Hopf algebra $\mathcal{K}_n$ via its natural action on the action groupoid $\mathbb{R}^n \rtimes \mathrm{Diff}(\mathbb{R}^n)^\delta$. This construction, which in principle parallels that of $\mathcal{H}_n$ (*cf.* [7, 16]) at the level of the prolongation groupoid on frame bundle, relies on the Lie-Hopf algebra techniques introduced in [21]. We refine the Lie-Hopf algebra decomposition of $\mathcal{K}_n$ in §2, by displaying a further bicrossed product factorization of its (commutative but not co-commutative) Hopf subalgebra $\mathcal{F}_{\mathcal{K}}$.

Using the full factorization we prove in §3 that the Hopf cyclic cohomology of $\mathcal{K}_n$ is isomorphic to the relative Hopf cyclic cohomology of the pair $(\mathcal{H}_n, \mathrm{GL}_n)$, and therefore (*cf.* [16]) to the truncated polynomial ring of Chern classes. A crucial ingredient for the proof is supplied by [21, Theorem 4.10], which provides the appropriate version of the van Est isomorphism for the present context.

The Chern-Weil type construction of cocycles representing the Hopf cyclic classes of $\mathcal{K}_n$ is performed in §4, by importing the method of [19, 20].

Finally, in §5 we illustrate the results of §4 in a concrete fashion, by producing completely explicit formulas for such cocycles in the case of $\mathcal{K}_1$. We conclude with a related calculation which brings in the Godbillon-Vey class. This serves to point out that, in order to incorporate the secondary Hopf cyclic transverse characteristic classes at the same direct level, it is necessary to

pass to a topological enhancement of the Hopf algebra $\mathcal{K}_n$. The latter will make the object of a forthcoming paper [18].

1 The Hopf algebra $\mathcal{K}_n$ and its standard action

Let $M = \mathbb{R}^n$ and let $\mathbf{G} := \mathrm{Diff}(M)^\delta$ be the group of all orientation preserving diffeomorphisms of M equipped with the discrete topology. The Hopf algebra $\mathcal{K}_n$ arises in the same way as $\mathcal{H}_n$ (*cf.* [7]), only at the level of the action groupoid $M \ltimes \mathbf{G}$ rather than of its frame bundle prolongation $M \ltimes \mathbf{G}$.
Thus, we consider the crossed product algebra $\mathcal{A} \equiv \mathcal{A}_{\mathbf{G}} := C_c^\infty(M) \rtimes \mathbf{G}$, where $\mathbf{G}$ acts on $C_c^\infty(M)$ by $\varphi \triangleright f = f \circ \varphi^{-1}$. A typical element of $\mathcal{A}_{\mathbf{G}}$ is a finite sum $\sum_i f_i U_{\varphi_i}^*$, where $f_i \in C_c^\infty(M)$ and $U_{\varphi_i}^*$ stands for $\varphi_i^{-1} \in \mathbf{G}$. The product of $\mathcal{A}_{\mathbf{G}}$ is determined by the multiplication rule

$$fU_\varphi^*\, gU_\psi^* = f \cdot (g \circ \varphi) U_{\psi\varphi}^*. \tag{1.1}$$

The vector fields $X_k \cong \partial_k := \frac{\partial}{\partial x^k}$ is made to act on $\mathcal{A}_{\mathbf{G}}$ by

$$X_k(fU_\varphi^*) = X_k(f) U_\varphi^* = \partial_k(f) U_\varphi^*. \tag{1.2}$$

One observes that

$$\begin{aligned} X_k(fU_\varphi^* \cdot gU_\psi^*) &= X_k(f \cdot (g \circ \varphi)) U_{\psi\varphi}^* \\ &= \partial_k(f) U_\varphi^* \cdot gU_\psi^* + \sum_i \partial_k(\varphi^i) \cdot fU_\varphi^*\, \partial_i(g) U_\psi^*, \end{aligned} \tag{1.3}$$

which proves that for all $a, b \in \mathcal{A}$ one has

$$X_k(ab) = X_k(a)b + \sum_i \sigma_k^i(a) X_i(b), \tag{1.4}$$

where

$$\sigma_j^i(fU_\varphi^*) = \partial_j(\varphi^i) fU_\varphi^*. \tag{1.5}$$

Another elementary manipulation gives

$$\sigma_j^i(fU_\varphi^* \cdot gU_\psi^*) = \sigma_j^i(f \cdot (g \circ \varphi)) U_{\psi\varphi}^* = \sum_k \partial_j(\varphi^k)\, fU_\varphi^* \cdot \partial_k(\psi^i)\, gU_\psi^*, \tag{1.6}$$

showing that for any $a,\ b \in \mathcal{A}$

$$\sigma^i_j(ab) = \sum_k \sigma^i_k(a)\ \sigma^k_j(b). \tag{1.7}$$

In addition, we introduce the Jacobian operator,

$$\sigma(fU^*_\varphi) = \det J(\phi) \cdot fU^*_\varphi, \tag{1.8}$$

where $J(\phi)(x) = \phi'(x)$ stands for the Jacobian matrix of ϕ at $x \in V$. It is an algebra automorphism of $\mathcal{A}$, whose inverse acts by

$$\sigma^{-1}(fU^*_\varphi) = \det J(\varphi^{-1}) \circ \varphi\ fU^*_\varphi = \det J(\varphi)^{-1}\ fU^*_\varphi. \tag{1.9}$$

Definition 1.1. *The algebra $\mathcal{K}_n$ is the unital subalgebra of $\mathcal{L}(\mathcal{A}_{\mathbf{G}})$ generated by the operators $\{X_k,\ \sigma^i_j,\ \sigma^{-p} \mid i,j,k = 1,\ldots,n;\ p \in \mathbb{N}\}$.*

Note that the algebra $\mathcal{K}_n$ also contains $\sigma = \sum_{\pi \in S_n} (-1)^\pi \sigma^1_{\pi(1)} \cdots \sigma^n_{\pi(n)}$, as well as the operators $\sigma^i_{j_1,\ldots,j_k}$, where $1 \leq i, j_1, \ldots, j_k \leq n$,

$$\sigma^i_{j_1,\ldots,j_k}(fU^*_\varphi) = \partial_{j_k} \cdots \partial_{j_1}(\varphi^i) \cdot fU^*_\varphi, \tag{1.10}$$

which are iteratively generated by the commutators

$$[X_\ell\ ,\ \sigma^i_{j_1,\ldots,j_k}] = \sigma^i_{j_1,\ldots,j_k,l}, \tag{1.11}$$

Other obvious relations are

$$[X_i, X_j] = 0, \tag{1.12}$$

$$\sigma^i_{j_1,\ldots,j_k} = \sigma^i_{j_{\pi(1)},\ldots,j_{\pi(k)}}, \quad \text{for any permutation } \pi \in S_k, \tag{1.13}$$

$$[\sigma^i_{j_1,\ldots,j_k}\ ,\ \sigma^p_{q_1,\ldots,q_m}] = 0, \qquad [\sigma^{-1}, \sigma^i_{j_1,\ldots,j_k}] = 0, \tag{1.14}$$

$$\sigma^{-1} \sum_{\pi \in S_n} (-1)^\pi \sigma^1_{\pi(1)} \cdots \sigma^n_{\pi(n)}\ = 1\,, \tag{1.15}$$

$$\begin{aligned}[X_k, \sigma^{-1}] = -\sigma^{-2} \sum_{\pi \in S_n} (-1)^\pi \big(\sigma^1_{\pi(1)k} \cdots \sigma^n_{\pi(n)} + \ldots \\ \ldots + \sigma^1_{\pi(1)} \cdots \sigma^n_{\pi(n)k}\big).\end{aligned} \tag{1.16}$$

Proposition 1.2. *The following collection of operators forms linear basis for $\mathcal{K}_n$:*

$$\sigma^{-p} \sigma^{i_1}_{J_1} \cdots \sigma^{i_m}_{J_m} X^{q_1}_{\ell_1} \cdots X^{q_k}_{\ell_k}\,; \tag{1.17}$$

here $1 \leq i_1, \ldots, i_m \leq n$, $1 \leq \ell_1, \ldots, \ell_k \leq n$, $p, q_1, \ldots, q_\ell \in \mathbb{Z}^+$, and J_p are finite ordered sets $J_p = \{j_1 \leq j_2 \leq \cdots \leq j_{m_p}\}$, with $1 \leq j_r \leq n$.

Proof. Similar to that of [16, Proposition 1.3] . ∎

We use the action of $\mathcal{K}_n$ on $\mathcal{A}_{\mathbf{G}}$ and the corresponding Leibnitz rules, such as (1.4), (1.5), to equip $\mathcal{K}_n$ with the bialgebra structure defined by the condition

$$\Delta(k) = k_{(1)} \otimes k_{(2)} \qquad \text{iff} \qquad k(ab) = k_{(1)}(a)k_{(2)}(b), \tag{1.18}$$

$$\varepsilon(k)1_{\mathcal{A}} = k(1). \tag{1.19}$$

In particular,

$$\Delta(X_\ell) = X_\ell \otimes 1 + \sigma_\ell^k \otimes X_k, \tag{1.20}$$

$$\Delta(\sigma_j^i) = \sigma_j^k \otimes \sigma_k^i, \tag{1.21}$$

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \Delta(\sigma^{-1}) = \sigma^{-1} \otimes \sigma^{-1}, \tag{1.22}$$

$$\Delta(\sigma_{j,k}^i) = \sigma_{j,k}^m \otimes \sigma_m^i + \sigma_k^\ell \sigma_j^m \otimes \sigma_{m,l}^i, \tag{1.23}$$

$$\Delta(\sigma_{j_1,\dots,j_k}^i) = [\Delta(X_{j_k}), \Delta(\sigma_{j_1,\dots,j_{k-1}}^i)], \tag{1.24}$$

$$\varepsilon(\sigma) = \varepsilon(\sigma^{-1}) = 1, \qquad \varepsilon(\sigma_j^i) = \delta_j^i, \qquad \varepsilon(X_\ell) = \varepsilon(\sigma_{j_1,\dots,j_k}^i) = 0. \tag{1.25}$$

Let $\mathcal{K}_{\mathrm{ab}}$ be the commutative polynomial algebra generated by $\sigma_{j_1,\dots,j_k}^i$ and σ^{-1}. It is obvious that $\mathcal{K}_{\mathrm{ab}}$ is a subbialgebra of $\mathcal{K}_n$.
For $k \in \mathcal{K}_{\mathrm{ab}}$ we define $\gamma_{\mathcal{K}}(k) : \mathbf{G} \to C^\infty(V)$ by

$$\gamma_{\mathcal{K}}(k)(\psi) = k(U_\psi^*)U_\psi. \tag{1.26}$$

One readily checks that the following cocycle property holds

$$\gamma_{\mathcal{K}}(k)(\psi_1\psi_2) = \gamma_{\mathcal{K}}(k_{(1)})(\psi_2)\gamma_{\mathcal{K}}(k_{(2)})(\psi_1) \circ \psi_2. \tag{1.27}$$

Using the above operators we then define the map $S : \mathcal{K}_{\mathrm{ab}} \to \mathcal{K}_{\mathrm{ab}}$ by

$$S(f)(gU_\psi^*) = \gamma_{\mathcal{K}}(f)(\psi^{-1}) \circ \psi \cdot g\, U_\psi^*. \tag{1.28}$$

Lemma 1.3. *The map S defined in* (1.28) *is the antipode of $\mathcal{K}_{\mathrm{ab}}$ and hence $\mathcal{K}_{\mathrm{ab}}$ is a Hopf algebra.*

Proof. We should show that S is the inverse of $\mathrm{Id}_{\mathcal{K}_{\mathrm{ab}}}$ in the convolution algebra $\mathrm{Hom}(\mathcal{K}_{\mathrm{ab}}, \mathcal{K}_{\mathrm{ab}})$. Indeed we first verify that S is the left inverse,

$$\begin{aligned}(\mathrm{Id}_{\mathcal{K}_{\mathrm{ab}}} * S)(f)(U_\psi^*) &= f_{(1)}S(f_{(2)})(U_\psi^*) = f_{(1)}(\gamma_{\mathcal{K}}(f_{(2)})(\psi^{-1}) \circ \psi\, U_\psi^*)\\ &= \gamma_{\mathcal{K}}(f_{(1)})(\psi)\gamma_{\mathcal{K}}(f_{(2)})(\psi^{-1}) \circ \psi\, U_\psi^* = \gamma_{\mathcal{K}}(f)(e)U_\psi^* = \varepsilon(f)U_\psi^*.\end{aligned} \tag{1.29}$$

Here in the last two equalities we have used the cocycle property (1.27) of $\gamma_{\mathcal{K}}$ and the very definition of ε. Similarly, one proves that S is a right convolution inverse to $\mathrm{Id}_{\mathcal{K}_{\mathrm{ab}}}$. ∎

To equip the algebra $\mathcal{K}_n$ itself with a Hopf algebra structure, we shall check that the Lie algebra V generated by the X_ℓ's together with the copposite Hopf algebra $\mathcal{F}_\mathcal{K} = {\mathcal{K}_{\rm ab}}^{\rm cop}$ form a Lie-Hopf pair in the sense of [21]. It will follow that the universal enveloping algebra $\mathcal{U}(V)$ of V together with $\mathcal{F}_\mathcal{K}$ form a matched pair of Hopf algebras (*cf.* [14]), from which we will reassemble $\mathcal{K}_n$. The Lie algebra V acts on $\mathcal{F}_\mathcal{K}$ from the left via

$$\triangleright : V \otimes \mathcal{F}_\mathcal{K} \to \mathcal{F}_\mathcal{K}, \qquad X \triangleright f = [X, f]. \tag{1.30}$$

Explicitly, for $f \in \mathcal{F}_\mathcal{K}$,

$$\begin{aligned} &(X_\ell \triangleright f)(gU_\psi^*) = (X_\ell f - f X_\ell)(gU_\psi^*) = X_\ell(\overset{\frown}{\gamma}_\mathcal{K}(f)gU_\psi^*) - f(\partial_\ell(g)U_\psi^*) \\ &= \partial_\ell(\overset{\frown}{\gamma}_\mathcal{K}(f)gU_\psi^* + \overset{\frown}{\gamma}_\mathcal{K}(f)\partial_\ell(g)U_\psi^* - \overset{\frown}{\gamma}_\mathcal{K}(f)(\psi)\partial_\ell(g)U_\psi^* \\ &= X_\ell(\overset{\frown}{\gamma}_\mathcal{K}(f))gU_\psi^*. \end{aligned} \tag{1.31}$$

We also define the following right coaction of $\mathcal{F}_\mathcal{K}$ on V by

$$\blacktriangledown : V \to V \otimes \mathcal{F}_\mathcal{K}, \blacktriangledown(X_\ell) = X_k \otimes \sigma_\ell^k\ . \tag{1.32}$$

Lemma 1.4. *Via the action and coaction defined in* (1.30) *and* (1.32), $\mathcal{F}_\mathcal{K}$ *is a* V*-Hopf algebra.*

Proof. Using the coproduct of σ_j^i, it is straightforward to see that (1.32) defines a coaction. We need to verify that the action $\triangleright$ and the coaction $\blacktriangledown$ satisfy the conditions required for a Lie-Hopf pair (*cf.* [21]).
First we should check that for any $g \in \mathcal{F}_\mathcal{K}$ and any $X \in V$ one has

$$\Delta(X \triangleright g) = X \bullet \Delta(g) = g_{(1)} \otimes X \triangleright g_{(2)} + X_{<0>} \triangleright g_{(1)} \otimes X_{<1>} g_{(2)}. \tag{1.33}$$

Indeed if $a, b \in \mathcal{A}$ and $f \in \mathcal{K}_{\rm ab}$ then

$$\begin{aligned} &\Delta(X_\ell \triangleright f)(ab) = [X_\ell, f](ab) = X_\ell f(ab) - f X_\ell(ab) = \\ &X_\ell(f_{(1)}(a) f_{(2)}(b)) - f(X_\ell(a)b + \sigma_\ell^k(a) X_k(b)) = \\ &X_\ell(f_{(1)}(a)) f_{(2)}(b) + \sigma_\ell^k(f_{(1)}(a)) X_k(f_{(2)}(b)) - \\ &f_{(1)}(X_\ell(a)) f_{(2)}(b) - f_{(1)}(\sigma_\ell^k(a)) f_{(2)}(X_k(b)) = \\ &[X_\ell, f_{(1)}](a) f_{(2)}(b) + \sigma_\ell^k f_{(1)}(a)[X_k, f_{(2)}](b) = \\ &X_\ell \triangleright f_{(1)}(a) f_{(2)}(b) + (X_{\ell_{<1>}} f_{(1)})(a)(X_{\ell_{<0>}} \triangleright f_{(2)})(b). \end{aligned} \tag{1.34}$$

Thus, for any $X \in V$,

$$\Delta(X \triangleright f) = X \triangleright f_{(1)} \otimes f_{(2)} + X_{<1>} f_{(1)} \otimes X_{<0>} \triangleright f_{(2)}. \tag{1.35}$$

Since the Lie algebra V is commutative and $\sigma^i_{j,k} = \sigma^i_{k,j}$ the coaction $\blacktriangledown$ satisfies the structure identity of V.
Finally, $\varepsilon(X_\ell \triangleright f) = 0$ for any $f \in \mathcal{K}_{\rm ab}$, which completes the verification of the axioms of a Lie-Hopf pair. $\square$

As a consequence, the bicrossed product Hopf algebra $\mathcal{F}_\mathcal{K} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(V)$ is well-defined. We define the map

$$\mathcal{I} : \mathcal{F}_\mathcal{K} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(V) \to \mathcal{K}_n^{\rm cop}, \qquad \mathcal{I}(f \blacktriangleright\!\!\vartriangleleft u) \ = \ fu. \tag{1.36}$$

Proposition 1.5. *The above map $\mathcal{I}$ is an isomorphism of bialgebras.*

Proof. One uses the linear basis (1.17) for $\mathcal{K}$ to see that $\mathcal{I}$ is an isomorphism of vector spaces. Let us check that the map $\mathcal{I}$ is an algebra homomorphism. We first use (1.31) to see that $uf = u_{(1)} \triangleright f u_{(2)}$ in $\mathcal{K}_n$,

$$\begin{aligned} &uf(gU^*_\psi) = u(\gamma_\mathcal{K}(f)(\psi)gU^*_\psi) = u(\gamma_\mathcal{K}(f)(\psi)g)U^*_\psi = \\ &u_{(1)}(\gamma_\mathcal{K}(f)(\psi))u_{(2)}(g)U^*_\psi = ((u_{(1)} \triangleright f)u_{(2)})(gU^*_\psi). \end{aligned} \tag{1.37}$$

This shows that $\mathcal{I}$ is indeed an algebra map.
To show that $\mathcal{I}$ is a coalgebra map it is necessary and sufficient to check that $\mathcal{I}$ commutes with coproduct on the generators. Indeed,

$$\begin{aligned} &(\mathcal{I} \otimes \mathcal{I})(\Delta_{\mathcal{F}_\mathcal{K} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(V)}(f \blacktriangleright\!\!\vartriangleleft 1)) = (\mathcal{I} \otimes \mathcal{I})(f_{(2)} \blacktriangleright\!\!\vartriangleleft 1) \otimes \mathcal{I}(f_{(1)} \blacktriangleright\!\!\vartriangleleft 1) \\ &= f_{(2)} \otimes f_{(1)} = \blacktriangledown_{\mathcal{K}_{\rm cop}}(f); \\ &(\mathcal{I} \otimes \mathcal{I})(\Delta_{\mathcal{F}_\mathcal{K} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(V)}(1 \blacktriangleright\!\!\vartriangleleft X_\ell)) = (\mathcal{I} \otimes \mathcal{I})(1 \blacktriangleright\!\!\vartriangleleft X_k \otimes \sigma^k_\ell \blacktriangleright\!\!\vartriangleleft 1 + 1 \blacktriangleright\!\!\vartriangleleft 1 \otimes 1 \blacktriangleright\!\!\vartriangleleft X_\ell) \\ &= X_k \otimes \sigma^k_\ell + 1 \otimes X_\ell = \blacktriangledown_{\mathcal{K}_{\rm cop}}(X_\ell). \end{aligned}$$

$\square$

Corollary 1.6. *The following defines the antipode of $\mathcal{K}_n$:*

$$S_\mathcal{K}(u) = S_{\mathcal{F}_\mathcal{K}}(u_{<1>})S_\mathcal{U}(u_{<0>}), \quad S_\mathcal{K}(f) = S_{\mathcal{F}_\mathcal{K}}(f). \tag{1.38}$$

Hence $\mathcal{K}_n$ is a Hopf algebra and $\mathcal{I}$ is an isomorphism of Hopf algebras.

Proof. One uses the antipode definition for a bicrossed product

$$S(f \blacktriangleright\!\!\triangleleft u) = (1 \blacktriangleright\!\!\triangleleft S(u_{<0>}))(S(fu_{<1>}) \blacktriangleright\!\!\triangleleft 1), \qquad f \in \mathcal{F},\, u \in \mathcal{U}. \tag{1.39}$$

and the fact that $\mathcal{U}$ is cocommutative and $\mathcal{F}_{\mathcal{K}}$ is commutative to see

$$S^{-1}_{\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\triangleleft \mathcal{U}}(f \blacktriangleright\!\!\triangleleft u) = S_{\mathcal{F}_{\mathcal{K}}}(fu_{<1>}) \blacktriangleright\!\!\triangleleft S_{\mathcal{U}}(u_{<0>}). \tag{1.40}$$

Since $\mathcal{I}$ is isomorphism of bialgebras and $\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\!\triangleleft \mathcal{U}$ is a Hopf algebra, $\mathcal{I}$ induces a unique antipode on $\mathcal{K}_n$. Equivalently, $S : \mathcal{K} \to \mathcal{K}$ is defined by the identity

$$S_{\mathcal{K}} = \mathcal{I} \circ S^{-1}_{\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\triangleleft \mathcal{U}} \circ \mathcal{I}^{-1}.$$

□

One sees that

$$S(\sigma^i_j) = \sigma^{-1} m^j_i, \tag{1.41}$$

where $m^p_q = (-1)^{p+q} \det M^p_q$, with M^p_q signifying the $(n-1) \times (n-1)$ matrix obtained by removing the qth row and pth column of the matrix $[\sigma^i_j]$. Also,

$$\begin{aligned} &S(\sigma) = \sigma^{-1}, \quad S(\sigma^{-1}) = \sigma, \\ &S(\sigma^i_{jk}) = -S(\sigma^i_r)S(\sigma^s_j)S(\sigma^t_k)\sigma^r_{st}, \\ &S(X_k) = -S(\sigma^\ell_k)X_\ell. \end{aligned} \tag{1.42}$$

2 Bicrossed product decomposition of $\mathcal{F}_{\mathcal{K}}$

We start with the decomposition of $\mathbf{G} = \mathbf{T} \cdot \mathbf{G}^\dagger$, where

$$\mathbf{T} = \{\varphi \in \mathbf{G} \mid \varphi(x) = x + b, \text{ for some } \; b \in \mathbb{R}^n\}, \tag{2.1}$$

$$\mathbf{G}^\dagger = \{\psi \in \mathbf{G} \mid \psi(0) = 0\}. \tag{2.2}$$

Any $\phi \in \mathbf{G}$ can be written uniquely as

$$\phi = \varphi_\phi \circ \psi_\phi, \qquad \varphi_\phi \in \mathbf{T}, \;\; \psi_\phi \in \mathbf{G}^\dagger. \tag{2.3}$$

where

$$\varphi_\phi(x) = x + \phi(0), \quad \psi_\phi(x) = \phi(x) - \phi(0). \tag{2.4}$$

This yields that $(\mathbf{T}, \mathbf{G}^\dagger)$ is a matched pair of groups with respect to the left action of $\mathbf{G}^\dagger$ on $\mathbf{T}$ and the right action of $\mathbf{T}$ on $\mathbf{G}^\dagger$ determined by

$$\psi \circ \varphi = (\psi \triangleright \varphi) \circ (\psi \triangleleft \varphi). \tag{2.5}$$

Thus, for $\varphi \in G^-$ defined by $\varphi(x) = x + b$, and for $\psi \in G^+$ one has

$$\psi \triangleright \varphi(x) = x + \psi(b), \qquad \psi \triangleleft \varphi(x) = \psi(x+b) - \psi(b). \tag{2.6}$$

The first equation shows that under the canonical identification of $\mathbb{R}^n$ with the translation group, $\varphi \in \mathbf{T} \leftrightarrow b = \varphi(0) \in \mathbb{R}^n$, the action of $\mathbf{G}^\dagger$ on $\mathbf{T}$ is just its natural action on $\mathbb{R}^n$.

Let $\mathcal{F}(\mathbf{G}^\dagger)$ be the commutative unital algebra of functions on $\mathbf{G}^\dagger$ generated by the coefficients of the Taylor expansion at 0,

$$\begin{aligned} &\beta^i_{j_1,\ldots,j_k}(\psi) = \partial_{j_k} \ldots \partial_{j_1} \psi^i(x)\Big|_{x=0}, \qquad 1 \leq i, j_1, \ldots, j_k \leq n, \ \ \psi \in \mathbf{G}^\dagger, \\ &\beta^{-1}(\psi) = \frac{1}{\det(\beta^i_j \psi)}. \end{aligned} \tag{2.7}$$

One proves as in [16, Proposition 2.5] that the group structure of $\mathbf{G}^\dagger$ induces a Hopf algebra structure on $\mathcal{F}(\mathbf{G}^\dagger)$, determined by

$$\Delta(f)(\psi_1, \psi_2) = f(\psi_1 \circ \psi_2), \qquad \psi_1, \psi_2 \in \mathbf{G}^\dagger, \tag{2.8}$$

$$S(f)(\psi) = f(\psi^{-1}), \qquad \psi \in \mathbf{G}^\dagger, \tag{2.9}$$

$$\varepsilon(f) = f(e). \tag{2.10}$$

One notes that for $\sigma^i_{j_1,\ldots,j_k} \in \mathcal{K}_n$ we have

$$\beta^i_{j_1,\ldots,j_k}(\psi) = \curlyvee(\sigma^i_{j_1,\ldots,j_k})(\psi)(0), \qquad \beta^{-1}(\psi) = \curlyvee(\sigma^{-1})(\psi)(0). \tag{2.11}$$

There is a unique isomorphism of Hopf algebras

$$\iota : {\mathcal{K}_{\mathrm{ab}}}^{\mathrm{cop}} \cong \mathcal{F}_{\mathcal{K}} \to \mathcal{F}(\mathbf{G}^\dagger), \tag{2.12}$$

with the property that

$$\iota(\sigma^{-1}) = \beta^{-1}, \quad \iota(\sigma^i_{j_1,\ldots,j_k}) = \beta^i_{j_1,\ldots,j_k}, \qquad 1 \leq i, j_1, \ldots, j_k \leq n, \ \ k \in \mathbb{N}. \tag{2.13}$$

One uses the right action of $\mathbf{T}$ on $\mathbf{G}^\dagger$ to get a left action of $\mathbf{T}$ on $\mathcal{F}(\mathbf{G}^\dagger)$ by

$$\varphi \triangleright f(\psi) = f(\psi \triangleleft \varphi). \tag{2.14}$$

We identify V with the Lie algebra of the Lie group $\mathbf{T}$, as the left invariant vector fields, and hence get the following action of V on $\mathcal{F}(\mathbf{T})$

$$(X \triangleright f)(\psi) = \frac{d}{dt}\Big|_{t=0} f(\psi \triangleleft \exp(tX)), \qquad f \in \mathcal{F}(\mathbf{G}^\dagger),\ \psi \in \mathbf{G}^\dagger\ X \in V. \tag{2.15}$$

To illustrate this action on the generators we compute:

$$\begin{aligned}&\partial_j(\psi \triangleleft \exp(tX_\ell))^i(x)\Big|_{x=0} = \\ &\partial_j(\psi^i(x+te^\ell) - \psi^i(te^\ell)) = (\partial_j\psi^i)(x+te^\ell)\Big|_{x=0} = (\partial_j\psi^i)(te^\ell)\end{aligned} \tag{2.16}$$

and continue by

$$\begin{aligned}&X_\ell \triangleright \beta^i_j(\psi) = \frac{d}{dt}\Big|_{t=0} \beta^i_j(\psi \triangleleft \exp(tX_\ell)) = \frac{d}{dt}\Big|_{t=0} (\partial_j\psi^i)(te^\ell) = \\ &\partial_\ell\partial_j\psi^i(x)\Big|_{x=0} = \beta^i_{j,l}(\psi).\end{aligned} \tag{2.17}$$

Similarly one proves that

$$X_\ell \triangleright \beta^i_{j_1,\dots,j_k} = \beta^i_{j_1,\dots,j_k,l}. \tag{2.18}$$

One observe that the action of $\mathbf{G}^\dagger$ on $\mathbf{T}$ is smooth and hence induces an action of $\mathbf{G}^\dagger$ on V,

$$\psi \triangleright X(g) = \frac{d}{dt}\Big|_{t=0} g(\psi \triangleright \exp(tX)), \qquad g \in C^\infty(\mathbb{R}^n). \tag{2.19}$$

In dual fashion, the action of $\mathbf{G}^\dagger$ on V defines a coaction

$$\blacktriangledown_V : V \to V \otimes \mathcal{F}(\mathbf{G}^\dagger), \tag{2.20}$$

defined by

$$\blacktriangledown_V(X_\ell) = X_j \otimes f^j_\ell, \qquad \text{if and only if } \ f^j_\ell(\psi)X_j = \psi \triangleright X_\ell \tag{2.21}$$

Let us explicitly compute this coaction. Since the action of $\mathbf{G}^{\dagger}$ on $\mathbf{T}$ is the natural one,

$$\psi \triangleright \exp(tX_\ell)(x) = x + \psi(te^\ell). \tag{2.22}$$

So for any $g \in C_c^\infty(\mathbb{R}^n)$ one has,

$$\begin{aligned} &(\psi \triangleright X_\ell)(g)(x) = \frac{d}{dt}\bigg|_{t=0} g(\psi \triangleright \exp(tX_\ell)) = \\ &\frac{d}{dt}\bigg|_{t=0} g(\psi(te^\ell)) = (\partial_k g)(x)\partial_\ell \psi^k(x)\bigg|_{x=0} = X_k(g)(x)\beta^i_j(\psi). \end{aligned} \tag{2.23}$$

We thus proved that

$$\blacktriangledown_V(X_\ell) = X_k \otimes \beta^k_\ell.$$

This show that (2.15), makes $\mathcal{F}(\mathbf{G}^\dagger)$ a $\mathcal{U}(V)$-module algebra and the map $\iota : \mathcal{F}_\mathcal{K} \to \mathcal{F}(\mathbf{G}^\dagger)$ is V-linear, where action of V on $\mathcal{F}_\mathcal{K}$ is defined by (1.30). Via the coaction defined by (2.21) $\mathcal{F}(\mathbf{G}^\dagger)$ is a V-Hopf algebra. The map ι induces the following isomorphism

$$\iota \blacktriangleright\!\!\triangleleft \operatorname{Id} : \mathcal{F}_\mathcal{K} \blacktriangleright\!\!\triangleleft \mathcal{U}(V) \to \mathcal{F}(\mathbf{G}^\dagger) \blacktriangleright\!\!\triangleleft \mathcal{U}(V), \tag{2.24}$$

Now we decompose the group $\mathbf{G}^\dagger$ into $\mathbf{G}^\dagger_0 \cdot \mathbf{N}$, where

$$\mathbf{G}^\dagger_0 = \{\psi \in \mathbf{G}^\dagger \mid \psi(x) = ax, \quad a \in \mathrm{GL}_n\}, \tag{2.25}$$

$$\mathbf{N} = \{\psi \in \mathbf{G}^\dagger \mid \psi'(0) = \mathrm{Id}\}. \tag{2.26}$$

Precisely, for any $\psi \in c$ we define $\lambda \in \mathbf{G}^\dagger_0$, and $\nu \in \mathbf{N}$ by

$$\lambda_\psi(x) = \psi'(0)x, \qquad \nu_\psi(x) = (\psi'(0))^{-1}\psi(x). \tag{2.27}$$

This unique decomposition determines the actions of $\mathbf{G}^\dagger_0$ on $\mathbf{N}$ and of $\mathbf{N}$ on $\mathbf{G}^\dagger_0$, by the prescription

$$\nu \circ \lambda = (\nu \triangleright \lambda) \circ (\nu \triangleleft \lambda), \tag{2.28}$$

for $\lambda \in \mathbf{G}^\dagger_0$ and $\nu \in \mathbf{N}$. More exactly, with $\lambda(x) = \mathbf{a} \cdot x, \;\; \mathbf{a} \in \mathrm{GL}_n(\mathbb{R})$,

$$\nu \triangleright \lambda = \lambda, \quad \text{and} \quad (\nu \triangleleft \lambda)(x) = \mathbf{a}^{-1}\nu(\mathbf{a} \cdot x), \tag{2.29}$$

reflecting the fact that $\mathbf{N}$ is a normal subgroup of $\mathbf{G}^\dagger$.

We let $\mathcal{F}(\mathbf{G}_0^\dagger)$ be the algebra generated by the functions

$$\alpha_j^i(\lambda) = \partial_j \lambda^j(x)\Big|_{x=0} = a_j^i\,, \quad \alpha^{-1}(\lambda) = \det(\mathbf{a}), \text{ for } \quad \lambda(x) = \mathbf{a}\cdot x, \tag{2.30}$$

i.e. the algebra $\mathcal{P}(\mathrm{GL}_n)$ of regular functions on $\mathrm{GL}_n(\mathbb{R})$.
Similarly, we let $\mathcal{F}(\mathbf{N})$ be the algebra generated by the restrictions to $\mathbf{N}$ of the Taylor coordinates (2.7) on $\mathbf{G}^\dagger$,

$$\alpha^i_{j_1,\ldots,j_k}(\nu) := \beta^i_{j_1,\ldots,j_k}(\nu) = \partial_{j_k}\cdots\partial_{j_1}\psi(x)\Big|_{x=0}, \qquad \nu\in\mathbf{N}. \tag{2.31}$$

Once again, $\mathcal{F}(\mathbf{G}_0^\dagger)$ and $\mathcal{F}(\mathbf{N})$ are in fact Hopf algebras with the usual structure,

$$\Delta(f)(\psi_1,\psi_2) = f(\psi_1\circ\psi_2), \quad \varepsilon(f) = f(e), \quad S(f)(\psi) = f(\psi^{-1}). \tag{2.32}$$

In particular

$$\Delta(\alpha_j^i) = \alpha_k^i\otimes\alpha_j^k, \quad \Delta(\alpha^{-1}) = \alpha^{-1}\otimes\alpha^{-1}, \tag{2.33}$$
$$\Delta(\alpha^i_{j,k}) = \alpha^i_{j,k}\otimes 1 + 1\otimes\alpha^i_{j,k} \tag{2.34}$$

Thus, the restriction maps of Hopf algebras

$$\begin{aligned}
&\pi_1:\mathcal{F}(\mathbf{G}^\dagger)\to\mathcal{F}(\mathbf{G}_0^\dagger), \qquad \pi_2:\mathcal{F}(\mathbf{G}^\dagger)\to\mathcal{F}(\mathbf{N}),\\
&\pi_1(\beta_j^i) = \alpha_j^i, \quad \pi_1(\beta^{-1}) = \alpha^{-1}, \quad \pi_1(\beta^i_{j_1,\ldots,j_k}) = 0,\\
&\pi_2(\beta_j^i) = \delta_j^i, \quad \pi_2(\beta^{-1}) = 1, \quad \pi_2(\beta^i_{j_1,\ldots,j_k}) = \alpha^i_{j_1,\ldots,j_k}
\end{aligned} \tag{2.35}$$

are maps of Hopf algebras. These projections admit as cross-sections the obvious inclusion maps $\mathcal{I}_i:\mathcal{F}(\mathbf{G}_i^+)\to\mathcal{F}(\mathbf{G}^\dagger)$, $i=1,2$.

$$\mathcal{I}_1(\alpha_j^i) = \beta_j^i, \quad \mathcal{I}_1(\alpha^{-1}) = \alpha^{-1}, \quad \mathcal{I}_2(\alpha^i_{j_1,\ldots,j_k}) = \beta^i_{j_1,\ldots,j_k}. \tag{2.36}$$

Lemma 2.1. *The map* $\blacktriangledown:\mathcal{F}(\mathbf{N})\to\mathcal{F}(\mathbf{N})\otimes\mathcal{F}(\mathbf{G}_0^\dagger)$ *defined by*

$$\blacktriangledown(f)(\nu,\lambda) = f(\nu\triangleleft\lambda), \tag{2.37}$$

is a coaction and makes $\mathcal{F}(\mathbf{N})$ *a* $\mathcal{F}(\mathbf{G}_0^\dagger)$ *a comodule Hopf algebra.*

Proof. Denoting the inclusion $\mathcal{F}(\mathbf{N}) \hookrightarrow \mathcal{F}(\mathbf{G}^{\dagger})$ by

$$\hat{f}(\psi) = f(\nu_{\psi}), \tag{2.38}$$

one observes that

$$\blacktriangledown(f)(\nu, \lambda) = f(\nu \triangleleft \lambda) = \hat{f}(\nu \circ \lambda) = \tag{2.39}$$

$$\hat{f}_{(1)}(\nu)\hat{f}_{(2)}(\lambda) = \pi_2(\hat{f}_{(1)}) \otimes \pi_1(\hat{f}_{(2)})(\nu, \lambda). \tag{2.40}$$

Since $\triangleleft$ is a group action, it is easily seen that $\blacktriangledown$ is a coaction. The fact that $\blacktriangledown$ preserves the product,

$$\blacktriangledown(f^1 f^2) = \blacktriangledown(f^1)\blacktriangledown(f^2),$$

is also clear.
Next we show that $\mathcal{F}(\mathbf{N})$ is $\mathcal{F}(\mathbf{G}_0^{\dagger})$-comodule coalgebra. Indeed,

$$\begin{aligned}
&f_{<0>}{}_{(1)} \otimes f_{<0>}{}_{(2)} \otimes f_{<1>}(\nu_1, \nu_2, \lambda) = \\
&f_{<0>} \otimes f_{<1>}(\nu_1 \circ \nu_2, \lambda) = f((\nu_1 \circ \nu_2) \triangleleft \lambda) = \\
&f((\nu_1 \triangleleft (\nu_2 \triangleright \lambda)) \circ (\nu_2 \triangleleft \lambda)) = \\
&f((\nu_1 \triangleleft \lambda) \circ (\nu_2 \triangleleft \lambda)) = \\
&f_{(1)}(\nu_1 \triangleleft \lambda) f_{(2)}(\nu_2 \triangleleft \lambda) = \\
&f_{(1)}{}_{<0>}(\nu_1) f_{(1)}{}_{<1>}(\lambda) f_{(2)}{}_{<0>}(\nu_2) f_{(2)}{}_{<1>}(\lambda) = \\
&f_{(1)}{}_{<0>} \otimes f_{(2)}{}_{<0>} \otimes f_{(1)}{}_{<1>} f_{(2)}{}_{<1>}(\nu_1, \nu_2, \lambda).
\end{aligned} \tag{2.41}$$

The last required condition is also satisfied:

$$\varepsilon(f_{<0>}) f_{<1>}(\lambda) = \blacktriangledown(f)(e, \lambda) = f(e \triangleleft \lambda) = f(e) = \varepsilon(f) 1_{\mathcal{F}(\mathbf{G}_1 G^+)}(\lambda). \tag{2.42}$$

$\square$

We note that, with the notation introduced above, the following identity holds:

$$\check{\alpha}_s^i \hat{\alpha}_{j_1,\dots,j_k}^s = \beta_{j_1,\dots,j_k}^i \ . \tag{2.43}$$

Since the action of $\mathcal{F}(\mathbf{N})$ on $\mathcal{F}(\mathbf{G}_0^{\dagger})$ is trivial, that is given by ε , we see that all conditions of matched pair of Hopf algebras are satisfied and we have the Hopf algebra $\mathcal{F}(\mathbf{G}_0^{\dagger}) \blacktriangleright\!\!\triangleleft \mathcal{F}(\mathbf{N})$. Moreover, as an algebra $\mathcal{F}(\mathbf{G}_0^{\dagger}) \blacktriangleright\!\!\triangleleft \mathcal{F}(\mathbf{N})$ is just $\mathcal{F}(\mathbf{G}_0^{\dagger}) \otimes \mathcal{F}(\mathbf{N})$ and as a coalgebra it is $\mathcal{F}(\mathbf{G}_0^{\dagger}) \blacktriangleright\!\!< \mathcal{F}(\mathbf{N})$. We will therefore adopt the latter notation.
There also is a natural left coaction of $\mathcal{F}(\mathbf{N})$ on $\mathcal{F}(\mathbf{G}_0^{\dagger})$, which we record below.

Lemma 2.2. *The map* $\blacktriangledown_L : \mathcal{F}(\mathbf{N}) \to \mathcal{F}(\mathbf{G}_0^\dagger) \otimes \mathcal{F}(\mathbf{N})$ *defined by*

$$\blacktriangledown_L(f)(\lambda, \nu) = \mathcal{I}_2(f)(\lambda, \nu). \tag{2.44}$$

defines a left coaction, which satisfies

$$\blacktriangledown_L(\alpha^i_{j_1,\ldots,j_k}) = \alpha^i_s \otimes \alpha^s_{j_1,\ldots,j_k}. \tag{2.45}$$

Proof. It suffices to prove that (2.45) holds, and this is straightforward:

$$\blacktriangledown_L(\alpha^i_{j_1,\ldots,j_k})(\lambda, \nu) = \beta^i_{j_1,\ldots,j_k}(\lambda \circ \nu) = \beta^i_s(\lambda)\beta^s_{j_1,\ldots,j_k}(\nu) = \alpha^i_s(\lambda)\alpha^s_{j_1,\ldots,j_k}(\nu).$$

□

We now define a natural map $\Phi : \mathcal{F}(\mathbf{G}^\dagger) \to \mathcal{F}(\mathbf{G}_0^\dagger) \otimes \mathcal{F}(\mathbf{N})$ by

$$\Phi(f)(\lambda, \nu) = f(\lambda \circ \nu) = \pi_1(f_{(1)}) \otimes \pi_2(f_{(2)})(\lambda, \nu). \tag{2.46}$$

Lemma 2.3. *The map* $\Phi : \mathcal{F}(\mathbf{G}^\dagger) \to \mathcal{F}(\mathbf{G}_0^\dagger) \otimes \mathcal{F}(\mathbf{N})$ *is an isomorphism of algebras.*

Proof. Φ is obviously linear, and

$$\Phi^{-1}(f^1 \otimes f^2) = \mathcal{I}_1(f^1)S(\mathcal{I}_1(f^2{}_{<-1>}))\mathcal{I}_2(f^2{}_{<0>}). \tag{2.47}$$

defines a two sided inverse for Φ. Indeed,

$$\begin{aligned}
&\Phi(\beta^i_j) = \alpha^i_j \otimes 1, \quad \Phi(\beta^{-1}) = \alpha^{-1} \otimes 1, \quad \Phi(\beta^i_{j_1,\ldots,j_k}) = \alpha^i_s \otimes \alpha^s_{j_1,\ldots,j_k},\\
&\Phi^{-1}(\alpha^i_j \otimes 1) = \beta^i_j, \quad \Phi^{-1}(\alpha^{-1} \otimes 1) = \beta^{-1}, \quad \Phi^{-1}(1 \otimes \beta^i_{j_1,\ldots j_k}) = S(\beta^i_s)\beta^s_{j_1,\ldots,j_k}.
\end{aligned} \tag{2.48}$$

Since both maps Φ and Φ^{-1} are algebra maps, the claim follows. □

Proposition 2.4. *The map* $\Phi : \mathcal{F}(\mathbf{G}^\dagger) \to \mathcal{F}(\mathbf{G}_0^\dagger) \otimes \mathcal{F}(\mathbf{N})$ *is an isomorphism of Hopf algebras.*

Proof. Both sides being commutative, it suffice to check the compatibility of Φ with the coalgebra structures.

Let $f \in \mathcal{F}(\mathbf{G}^\dagger)$ be of the form $f = f^1 f^2$, with $f_1 \in \mathcal{F}(\mathbf{G}_0^\dagger)$, $f_2 \in \mathcal{F}(\mathbf{N})$, which means that $\Phi(f) = f^1 \blacktriangleright\!\!< f^2$. The comultiplication of $\mathcal{F}(\mathbf{G}_0^\dagger) \blacktriangleright\!\!< \mathcal{F}(\mathbf{N})$ is given by

$$\begin{aligned}
&\Delta_{\mathcal{F}_1(\mathbf{G}^\dagger)\blacktriangleright\!\!<\mathcal{F}_2(\mathbf{G}^\dagger)}(\Phi(f))(\lambda_1,\nu_1;\lambda_2,\nu_2)\\
&= \left(f^1{}_{(1)} \blacktriangleright\!\!< f^2{}_{(1)}{}_{<0>} \otimes f^1{}_{(2)} f^2{}_{(1)}{}_{<0>} \blacktriangleright\!\!< f^2{}_{(2)}\right)(\lambda_1,\nu_1;\lambda_2,\nu_2)\\
&= f^1{}_{(1)}(\lambda_1) f^2{}_{(1)}{}_{<0>}(\nu_1) f^1{}_{(2)}(\lambda_2) f^2{}_{(1)}{}_{<0>}(\lambda_2) f^2{}_{(2)}(\nu_2) =\\
&= (f^1{}_{(1)}(\lambda_1) f^1{}_{(2)}(\lambda_2))\ (f^2{}_{(1)}{}_{<0>}(\lambda_2) f^2{}_{(1)}{}_{<0>}(\nu_1))\ f^2{}_{(2)}(\nu_2)\\
&= f^1(\lambda_1 \circ \lambda_2) f^2{}_{(1)}(\nu_1 \triangleleft \lambda_2) f^2{}_{(2)}(\nu_2)\\
&= f^1(\lambda_1 \circ \lambda_2) f^2((\nu_1 \triangleleft \lambda_2) \circ \nu_2).
\end{aligned} \tag{2.49}$$

On the other hand one uses (2.28) and the fact that $\nu \triangleright \lambda = \lambda$ for any $\lambda \in \mathbf{G}_0^\dagger$ and any $\nu \in G_2^+$ to see that

$$\begin{aligned}
&\Phi \otimes \Phi(\Delta(f))(\lambda_1,\nu_1;\lambda_2,\nu_2)\\
&= \Phi(f_{(1)})(\lambda_1,\lambda_2)\Phi(f_{(2)})(\lambda_2,\nu_2))\\
&= f_{(1)}(\lambda_1 \circ \nu_1) f_{(2)}(\lambda_2 \circ \nu_2)\\
&= f(\lambda_1 \circ \nu_1 \circ \lambda_2 \circ \nu_2)\\
&= f(\lambda_1 \circ \lambda_2 \circ (\nu_2 \triangleleft \lambda_1) \circ \nu_2)\\
&= f^1(\lambda_1 \circ \lambda_2) f^2(\nu_2 \triangleleft \lambda_1) \circ \nu_2).
\end{aligned} \tag{2.50}$$

Finally it is easy to see that

$$\varepsilon_{\mathcal{F}_1(\mathbf{G}^\dagger)\blacktriangleright\!\!<\mathcal{F}_2(\mathbf{G}^\dagger)}(\Phi(f)) = f^1(e) f^2(e) = f(e) = \varepsilon_{\mathcal{F}(\mathbf{G}^\dagger)}(f). \tag{2.51}$$

$\square$

Via the above isomorphism we identify

$$\beta^i_j = \alpha^i_j \blacktriangleright\!\!< 1, \quad \beta^{-1} = \alpha^{-1} \blacktriangleright\!\!< 1, \quad \beta^i_{j_1,\ldots j_k} = \alpha^i_s \blacktriangleright\!\!< \alpha^s_{j_1,\ldots,j_k} \tag{2.52}$$

For $\alpha^i_{j_1,\ldots,j_k}$ we have

$$\blacktriangledown(\alpha^i_{j_1,\ldots,j_k}) = \alpha^r_{s_1,\ldots,s_k} \otimes S(\alpha^i_r)\alpha^{s_1}_{j_1} \cdots \alpha^{s_k}_{j_k}. \tag{2.53}$$

Let $\mathfrak{gl}_n$ be the Lie algebra $\mathrm{GL}_n(\mathbb{R})$. One defines a Hopf pairing between $\mathcal{F}(\mathbf{G}_0^\dagger)$ and $\mathcal{U}(\mathfrak{gl}_n)$ by extending the natural pairing

$$\langle f, Y\rangle = Y(f) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(tY)), \qquad Y \in \mathfrak{gl}_n, \lambda \in \mathbf{G}_0^\dagger. \tag{2.54}$$

This means that

$$\begin{aligned}&\langle f^1 f^2, u\rangle = \langle f^1, u_{(1)}\rangle\langle f^2, u_{(2)}\rangle, \quad \langle f, u^1u^2\rangle = \langle f_{(1)}, u^1\rangle\langle f_{(2)}, u^2\rangle,\\ &\langle f, 1\rangle = \varepsilon(f), \qquad \langle 1, u\rangle = \varepsilon(u), \qquad \langle f, S(u)\rangle = \langle S(f), u\rangle.\end{aligned} \tag{2.55}$$

We now define the action $\mathfrak{gl}_n \otimes \mathcal{F}(\mathbf{N}) \to \mathcal{F}(\mathbf{N})$ by

$$Y \triangleright f = f_{<0>} Y(f_{<1>}). \tag{2.56}$$

We denote the standard basis of $\mathfrak{gl}_n$ by Y_j^i, $1 \le i, j \le n$. One first observes that $Y_j^i(\alpha_q^p) = \delta_q^i \delta_j^p$. Then we use the Hopf pairing properties (2.55) to see that

$$Y_j^i \triangleright \alpha^p_{q_1,\dots,q_m} = \sum_s \delta^i_{q_s} \alpha^p_{q_1,\dots,j,\dots q_m} - \delta^p_j \alpha^i_{q_1,\dots,q_m}, \tag{2.57}$$

By restricting the action of $\mathbf{G}^\dagger$ on $\mathbf{T}$ to $\mathbf{G}_0^\dagger$ we get the coaction

$$\begin{aligned}&V \to V \otimes \mathcal{F}(\mathbf{G}_0^\dagger),\\ &X_k \to X_s \otimes \alpha_k^s.\end{aligned} \tag{2.58}$$

We use this coaction to define an action of $\mathfrak{gl}_n$ on V via

$$Y \triangleright X = \langle X_{<0>}, Y\rangle X_{<1>}. \tag{2.59}$$

Note that the action of Y_j^i on X_k is indeed the natural action of $\mathfrak{gl}_n$ on $\mathbb{R}^n$, *i.e.*

$$Y_j^i \triangleright X_k \;=\; \delta_k^i\, X_j. \tag{2.60}$$

3 Hopf cyclic cohomology of $\mathcal{K}_n$

For the reader's convenience we recall two basic notions.
Given a Hopf algebra $\mathcal{H}$, a character $\delta : \mathcal{H} \to \mathbb{C}$ and a group-like element $\sigma \in \mathcal{H}$, the pair (δ, σ) is called a *modular pair in involution* if

$$\delta(\sigma) = 1, \quad \text{and} \quad S_\delta^2 = Ad_\sigma, \tag{3.1}$$

where $Ad_\sigma(h) = \sigma h \sigma^{-1}$ and $S_\delta(h) = \delta(h_{(1)})S(h_{(2)})$, $h \in \mathcal{H}$. To such a datum was associated in [9] a cyclic module whose cohomology, called Hopf cyclic, is denoted $HC^\bullet(\mathcal{H}; {}^\sigma\mathbb{C}_\delta)$.

Let now $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\mathcal{F}$ be a $\mathfrak{g}$-Hopf algebra (*cf.* [21]), on which $\mathfrak{g}$ coacts via $\blacktriangledown_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$. The modular character of $\delta : \mathfrak{g} \to \mathbb{C}$,

$$\delta(X) = \mathrm{Trace}(\mathrm{ad}_X), \qquad X \in \mathfrak{g},$$

extends to a character of $\mathcal{U}(\mathfrak{g})$. One then further extends δ to a character of $\mathcal{F} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{g})$ by

$$\delta(f \blacktriangleright\!\!\vartriangleleft u) = \varepsilon(f)\delta(u).$$

In a dual fashion, one defines a group-like element in $\mathcal{F}$ as follows. The (first-order) matrix coefficients $f^i_j \in \mathcal{F}$ of the coaction $\blacktriangledown_{\mathfrak{g}}$ are given by the equation

$$\blacktriangledown_{\mathfrak{g}}(X_j) = \sum_{i=1}^{n} X_i \otimes f^i_j, \qquad n = \dim \mathfrak{g};$$

they satisfy the relation

$$\Delta(f^j_i) = \sum_{k=1}^{n} f^j_k \otimes f^k_i .$$

$$\sigma_F := \det(f^i_j) = \sum_{\pi \in S_n} (-1)^{\pi} f^1_{\pi(1)} \cdots f^n_{\pi(n)}. \tag{3.2}$$

One then defines a group-like element in $F \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{g})$ by setting

$$\sigma := \sigma_F \blacktriangleright\!\!\vartriangleleft 1, \quad \text{where} \quad \sigma_F := \sum_{\pi \in S_n} (-1)^{\pi} f^1_{\pi(1)} \cdots f^n_{\pi(n)}.$$

It is shown in [21, Theorem 3.2] that the (δ, σ) defines a modular pair in involution for the Hopf algebra $F \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{g})$.

We now return to the V-Hopf algebra $\mathcal{F}_{\mathcal{K}}$ of §1, equipped with the action and the coaction defined in (2.15) and (2.20). Since the Lie algebra V is commutative, δ coincides with ε and hence (ε, σ) is a modular pair in involution for $\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(V)$. Denoting by ${}^{\sigma}\mathbb{C}$ the one-dimensional left comodule and right module over $\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}$ determined by the group-like σ and the character ε respectively, we are thus in a position to form the Hopf cyclic cohomology $HC^{\bullet}(\mathcal{F}_{\mathcal{K}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}, {}^{\sigma}\mathbb{C})$ of the canonically associated (b, B)-bicomplex (*cf.* [7, 8, 9]).
In order to compute this cohomology, we employ a quasi-isomorphic bicomplex, which takes advantage of the bicrossed product structure of the Hopf

algebra $\mathcal{K}_n$ as well as of the particular nature of its components. Referring the reader to [16, 17] for details concerning the intermediate steps, we proceed to describe the resulting quasi-isomorphic bicomplex.
The Lie algebra V admits the following right action on ${\mathcal{F}_{\mathcal{K}}}^{\otimes q}$

$$X \bullet (f^1 \otimes \cdots \otimes f^q) = \tag{3.3}$$
$$X_{(1)}{}_{<0>} \triangleright f^1 \otimes X_{(1)}{}_{<1>}(X_{(2)}{}_{<0>} \triangleright f^2) \otimes \cdots \otimes X_{(1)}{}_{<q-1>} \dots X_{(q-1)}{}_{<1>}(X_{(q)} \triangleright f^q),$$

On the other hand, since $\mathcal{F}_{\mathcal{K}}$ is commutative, the coaction $\blacktriangledown : V \to V \otimes \mathcal{F}_{\mathcal{K}}$, extends from V to a unique coaction $\blacktriangledown_V : \wedge^p V \to \wedge^p V \otimes \mathcal{F}_{\mathcal{K}}$. After tensoring it with the right coaction of ${}^\sigma\mathbb{C}$ we obtain the coaction

$$\begin{aligned} &\blacktriangledown_{{}^\sigma\mathbb{C}\otimes\wedge V}(\mathbf{1} \otimes X^1 \wedge \cdots \wedge X^q) = \\ &\mathbf{1} \otimes X^1{}_{<0>} \wedge \cdots \wedge X^q{}_{<0>} \otimes \sigma^{-1} X^1{}_{<1>} \dots X^q{}_{<1>}. \end{aligned} \tag{3.4}$$

Let $\{X_i\}_{1\leq i\leq n}$ be the basis for V and let $\{\theta^j\}_{1\leq j\leq n}$ denote the dual basis of V^*. One defines the dual left coaction on $\blacktriangledown_V^* : V^* \to V^* \otimes \mathcal{F}_{\mathcal{K}}$ by

$$\blacktriangledown_V^*(\theta^i) = \sum_j \beta_j^i \otimes \theta^j, \quad \text{where} \quad \blacktriangledown_V(X_i) = X_j \otimes \beta_i^j. \tag{3.5}$$

We extend this coaction on $\wedge^\bullet V^*$ diagonally and observe that the result is a left coaction just because $\mathcal{F}_{\mathcal{K}}$ is commutative. For later use we record below that if $\omega := \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$ then

$$\omega_{<-1>} \otimes \omega_{<0>} = \sum_{1\leq l_j\leq m} f_{l_1}^{i_1} \cdots f_{l_k}^{i_k} \otimes \theta^{l_1} \wedge \cdots \wedge \theta^{l_k}. \tag{3.6}$$

One uses the antipode of $\mathcal{F}_{\mathcal{K}}$ to turn it into a right coaction $\blacktriangledown_{\wedge V^*} : \wedge^p V^* \to \wedge^p V^* \otimes \mathcal{F}_{\mathcal{K}}$, as follows:

$$\blacktriangledown_{\wedge V^*}(\omega) = \omega_{<1>} \otimes S(\omega_{<-1>}). \tag{3.7}$$

We now use the above ingredients to build the following bicomplex:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\wedge^2 V^* \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \wedge^2 V^* \otimes \mathcal{F}_{\mathcal{K}} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \wedge^2 V^* \otimes {\mathcal{F}_{\mathcal{K}}}^{\otimes 2} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \dots \\
V^* \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & V^* \otimes \mathcal{F}_{\mathcal{K}} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & V^* \otimes {\mathcal{F}_{\mathcal{K}}}^{\otimes 2} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \dots \\
\mathbb{C} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \mathcal{F}_{\mathcal{K}} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & {\mathcal{F}_{\mathcal{K}}}^{\otimes 2} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_{\mathcal{K}}}"] & \dots
\end{tikzcd}
\quad . \tag{3.8}
$$

The vertical coboundary $\partial_{V^*} : C^{p,q} \to C^{p,q+1}$ is the Lie algebra cohomology coboundary of the Lie algebra V with coefficients in ${\mathcal{F}_{\mathcal{K}}}^{\otimes p}$, where the action of V is given by

$$
(\mathbf{1} \otimes f^1 \otimes \cdots \otimes f^p) \blacktriangleleft X = - \mathbf{1} \otimes X \bullet (f^1 \otimes \cdots \otimes f^p). \tag{3.9}
$$

The horizontal b-coboundary $b^*_{\mathcal{F}_{\mathcal{K}}}$ has the expression

$$
\begin{aligned}
&b^*_{\mathcal{F}_{\mathcal{K}}}(\alpha \otimes f^1 \otimes \cdots \otimes f^q) = \\
&\alpha \otimes 1 \otimes f^1 \otimes \cdots \otimes f^q + \sum_{i=1}^{q} (-1)^i \alpha \otimes f^1 \otimes \cdots \otimes \Delta(f^i) \otimes \cdots \otimes f^q + \\
&(-1)^{q+1} \otimes \alpha_{<1>} \otimes f^1 \otimes \cdots \otimes f^q \otimes \alpha_{<-1>}.
\end{aligned} \tag{3.10}
$$

At this stage we recall that by Proposition (2.4) the Hopf subalgebra has a further factorization,

$$
\mathcal{F}_{\mathcal{K}} \cong \mathcal{F}(\mathbf{G}_0^{\dagger}) \blacktriangleright\!\!< \mathcal{F}(\mathbf{N}). \tag{3.11}
$$

This allows to apply the same treatment alluded to above to each row of the bicomplex (3.8).
Let us describe the bicomplex which computes the cohomology of the pth row. To simplify the notation, in what follows we abbreviate $\mathcal{F}_1 := \mathcal{F}(\mathbf{G}_0^{\dagger})$ and $\mathcal{F}_2 := \mathcal{F}(\mathbf{N})$.

Diagrammatically,

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\wedge^p V^* \otimes \mathcal{F}_2^{\otimes 2} \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_1}"] & \wedge^p V^* \otimes \mathcal{F}_2^{\otimes 2} \otimes \mathcal{F}_1 \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \wedge^p V^* \otimes \mathcal{F}_2^{\otimes 2} \otimes \mathcal{F}_1^{\otimes 2} \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \dots \\
\wedge^p V^* \otimes \mathcal{F}_2 \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \wedge^p V^* \otimes \mathcal{F}_2 \otimes \mathcal{F}_1 \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \wedge^p V^* \otimes \mathcal{F}_2 \otimes \mathcal{F}_1^{\otimes 2} \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \dots \\
\wedge^p V^* \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \wedge^p V^* \otimes \mathcal{F}_1 \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \wedge^p V^* \otimes \mathcal{F}_1^{\otimes 2} \arrow[r, "b^*_{\mathcal{F}_1}"] \arrow[u, "b^*_{\mathcal{F}_2}"] & \dots
\end{tikzcd}
\quad . \tag{3.12}
$$

The qth row above is the Hochschild complex of the coalgebra $\mathcal{F}_1$ with coefficients in the comodule $\wedge^p V^* \otimes \mathcal{F}_2^{\otimes q}$ defined by

$$
\begin{gathered}
\blacktriangledown_{V^* \otimes \mathcal{F}_2} : \wedge^q V^* \otimes \mathcal{F}_2^{\otimes \bullet} \to \mathcal{F}_1 \otimes \wedge^q V^* \otimes \mathcal{F}_2^{\otimes \bullet}, \\
\blacktriangledown(\omega \otimes \tilde{f}) = \omega_{<-1>} S(\tilde{f}_{<1>}) \otimes \omega_{<0>} \otimes \tilde{f}_{<0>},
\end{gathered}
\tag{3.13}
$$

where we use the natural left coaction of $\mathcal{F}_1$ on $\wedge^p V^*$ defined in (3.6).
In the above bicomplex we also use the coaction of $\mathcal{F}_1$ on $\mathcal{F}_2$ defined in (2.37) and extend it on $\tilde{f} = f^1 \otimes \cdots \otimes f^q \in \mathcal{F}_2^{\otimes q}$ by

$$
\tilde{f}_{<0>} \otimes \tilde{f}_{<1>} = f^1{}_{<0>} \otimes \cdots \otimes f^q{}_{<0>} \otimes f^1{}_{<1>} \cdots f^q{}_{<1>}, \tag{3.14}
$$

The columns of the bicomplex are just the Hochschild complexes of the coalgebra $\mathcal{F}_2$ with trivial coefficients $\wedge^p V^* \otimes \mathcal{F}_1^{\otimes \bullet}$.

Proposition 3.1. *The cohomology of the qth row of the bicomplex* (3.12) *is concentrated in the first column and coincides with* $(\wedge^p V^* \otimes \mathcal{F}_2^{\otimes q})^{\mathfrak{gl}_n}$.

Proof. The Lie algebra $\mathfrak{gl}_n$, viewed as a subalgebra of formal vector fields on $\mathbb{R}^n$, acts naturally on both V^* and on $\mathcal{F}_2$; it is this standard action which appears in the above statement.
Set $Z^{p,q} = \wedge^p V^* \otimes \mathcal{F}_2^{\otimes q}$. The qth row is the Hochschild complex of the $\mathcal{F}_1$ with coefficients in $Z^{p,q}$. We use the identification of $\mathcal{F}_1$ with $\mathcal{P}(\mathrm{GL}_n)$ and, since GL_n is reductive, we are in a position to apply [21, Theorem 4.8] to

infer that the cohomology of the row is concentrated in the 0th cohomology group, in other words that

$$H^k(\mathcal{F}_1, Z^{p,q}) \;=\; 0, \quad k>0, \quad H^0(\mathcal{F}_1\;,\; Z^{p,q}) \;=\; (Z^{p,q})^{\mathcal{F}_1}. \tag{3.15}$$

Here $(Z^{p,q})^{\mathcal{F}_1}$ is the space of coinvariants elements with respect to the coaction $\mathcal{F}_1$ on $Z^{p,q}$ defined in (3.7), i.e.,

$$(Z_{p,q})^{\mathcal{F}_1} = \{\omega \otimes \tilde{f} \;\mid\; \blacktriangledown_{V^*\otimes\mathcal{F}_2}(\omega\otimes\tilde{f}) = 1\otimes\omega\otimes\tilde{f}\} \tag{3.16}$$

The action of $\mathfrak{gl}_n$ on $\mathcal{F}_2$ is that defined in (2.57). The action of $\mathfrak{gl}_n$ on V^* comes from the coaction of $\mathcal{F}_1$ on V^* defined by (3.5). Explicitly,,

$$\theta^k \triangleleft Y^i_j \;=\; \langle \beta^k_\ell\;,\; Y^i_j\rangle\, \theta^\ell \;=\; \delta^k_j\delta^i_\ell\, \theta^\ell \;=\; \delta^k_j\, \theta^i, \tag{3.17}$$

which is transpose of the standard action of $\mathfrak{gl}_n$ on V defined in (2.59).
Let us show that this space of coinvariants coincides with the invariants under $\mathfrak{gl}_n$. The right action of $\mathfrak{gl}_n$ on $Z^{p,q}$ is given by

$$(\omega\otimes\tilde{f})\triangleleft Y = \langle(\omega\otimes\tilde{f})_{<-1>}\;,\; Y\rangle\,(\omega\otimes\tilde{f})_{<0>}, \tag{3.18}$$

and we need to check that $\omega\otimes\tilde{f}\in(Z^{p,q})^{\mathcal{F}_1}$ if and only if $\omega\otimes\tilde{f}\in(Z^{p,q})^{\mathfrak{gl}_n}$, or equivalently,

$$\omega\otimes\tilde{f}\in(Z^{p,q})^{\mathcal{F}_1} \iff \omega\triangleleft Y\otimes\tilde{f} - \omega\otimes Y\triangleright\tilde{f} = 0. \tag{3.19}$$

$\square$

Theorem 3.2. *The periodic Hopf cyclic cohomology $HP^\bullet(\mathcal{K}_n;\, {}^{\sigma^{-1}}\mathbb{C})$ is isomorphic to the truncated ring of Chern classes $P_{2n}[c_1,\ldots,c_n]$.*

Proof. By Proposition 3.1 the total cohomology of the bicomplex (3.12) reduces to the cohomology of the following complex

$$(\wedge^p V^*)^{\mathfrak{gl}_n} \xrightarrow{\;b^*_{\mathcal{F}_2}\;} (\wedge^p V^*\otimes\mathcal{F}_2)^{\mathfrak{gl}_n} \xrightarrow{\;b^*_{\mathcal{F}_2}\;} \cdots \tag{3.20}$$

We still need to calculate the total cohomology of (3.8). Via the above identification, that bicomplex is quaisi-isomorphic with the following one,

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(\wedge^2 V^*)^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & (\wedge^2 V^* \otimes \mathcal{F}_2)^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & (\wedge^2 V^* \otimes {\mathcal{F}_2}^{\otimes 2})^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & \dots \\
(V^*)^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & (V^* \otimes \mathcal{F}_2)^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & (V^* \otimes {\mathcal{F}_2}^{\otimes 2})^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & \dots \\
\mathbb{C} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & (\mathcal{F}_2)^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & ({\mathcal{F}_2}^{\otimes 2})^{\mathfrak{gl}_n} \arrow[u, "\partial_{V^*}"] \arrow[r, "b^*_{\mathcal{F}_2}"] & \dots
\end{tikzcd} \quad . \tag{3.21}
$$

One uses (2.2), (2.26), and (2.31) on one hand and [16, Proposition 2.1, Definition 2.4] on the other hand to observe that

$$\mathcal{F}_2 = \mathcal{F}_{\mathcal{H}}\,, \tag{3.22}$$

where $\mathcal{F}_{\mathcal{H}} \subset \mathcal{H}_n$ is the Hopf subalgebra, denoted by $\mathcal{F}(\mathbf{N})$ in [16], such that $\mathcal{H}_n^{\rm cop} = \mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{gl}_n^{\rm aff})$.
After this identification one applies (2.56), (2.58), and (2.60) to observe that the bicomplex (3.21) is identified with the bicomplex (4.12) in [17] for $\mathfrak{h} = \mathfrak{gl}_n$, or alternately with the bicomplex (3.42) in [16]. The total cohomology of the latter bicomplex is computed in [16, Theorem 3.25], and shown to be isomorphic to $P_{2n}[c_1, \dots, c_n]$. $\square$

There is an alternative way to formulate the above result, which relies on identifying, as coalgebras, $\mathcal{K}_n$ and the quotient coalgebra $\mathcal{Q}_n := \mathcal{H}_n \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C}$. First, one identifies the copposite coalgebra $\mathcal{Q}_n^{\rm cop}$ to $\mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{gl}_n^{\rm aff}) \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C}$ as the $\mathcal{U}(\mathfrak{gl}_n)$, is isomorphic to the crossed product coalgebras $\mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!< \mathcal{U}(V)$, via the map

$$
\begin{aligned}
\kappa_{\mathcal{H}} : \mathcal{H}_n^{\rm cop} \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C} = (\mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!\vartriangleleft \mathcal{U}(\mathfrak{gl}_n^{\rm aff})) \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C} \to \mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!< \mathcal{U}(V) \\
\kappa_{\mathcal{H}}(f \blacktriangleright\!\!\vartriangleleft XY \otimes_{\mathcal{U}(\mathfrak{gl}_n)} 1) = \varepsilon(Y) f \blacktriangleright\!\!< X;
\end{aligned} \tag{3.23}
$$

here $\mathcal{F}_{\mathcal{H}}$ coacts on $\mathcal{U}(V)$ via its coaction on $\mathcal{U}(\mathfrak{gl}_n^{\rm aff})$ followed by the projection $\pi : \mathcal{U}(\mathfrak{gl}_n^{\rm aff}) \to \mathcal{U}(V)$ that is defined by $\pi(XY) = \varepsilon(Y)X$, and is a map of coalgebras. It is clear that $\kappa_{\mathcal{H}}$ is an isomorphism.
We next consider the map $\kappa^\dagger : \mathcal{F}(\mathbf{G}^\dagger) \blacktriangleright\!\!< \mathcal{U}(V) \to \mathcal{F}_{\mathcal{H}} \blacktriangleright\!\!< \mathcal{U}(V)$ by the formula

$$\kappa^\dagger(\beta^i_{j_1,j_2,\ldots j_s} \blacktriangleright\!\!< u) = \delta^i_s \alpha^s_{j_1,j_2,\ldots,j_s} \blacktriangleright\!\!< u, \tag{3.24}$$

where δ^i_j is the Kronecker's delta tensor and $\alpha^i_j := \delta^i_j$. This map is quite natural, being the same as $r_{\mathcal{H}} \otimes \mathrm{Id}$, where $r_{\mathcal{H}} : \mathcal{F}(\mathbf{G}^\dagger) \to \mathcal{F}(\mathbf{N}) \cong \mathcal{F}_{\mathcal{H}}$ is the restriction map, dual to the inclusion $\mathbf{N} \hookrightarrow \mathbf{G}^\dagger$.

Theorem 3.3. *The map $\kappa := \kappa_{\mathcal{H}}^{-1} \circ \kappa^\dagger : \mathcal{F}(\mathbf{G}^\dagger) \blacktriangleright\!\!< \mathcal{U}(V) \to \mathcal{H}_n^{\rm cop} \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C}$ is a morphism of coalgebras which induces a quasi-isomorphism of Hochschild cohomology complexes*

$$\{ {}^{\sigma}\mathbb{C} \otimes \mathcal{K}^{{\rm cop}\,\otimes \bullet}, b\} \to \{\mathbb{C}_\delta \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathcal{Q}^{{\rm cop}\,\otimes \bullet}, b\}.$$

This in turn yields an isomorphism $HC^*(\mathcal{K};\, {}^{\sigma^{-1}}\mathbb{C}) \cong HC(\mathcal{H}_n, GL_n; \mathbb{C}_\delta)$.

Proof. Proposition 2.4 guarantees that $\kappa^\dagger$ is a map of coalgebras. Using the definition (3.24), which in particular implies that $\kappa^\dagger(\beta^i_j) = \delta^i_j$, it is easy to check that κ induces a chain map at the level of Hochchild complexes.
The second claim follows by combining the following two facts: the vanishing of the Connes boundary map B at the level of both Hochschild complexes; the vanishing of the Hochschild cohomology groups outside degree n of both sides, ensured by Theorem 3.2, resp. [16, Theorem 3.25]. ☐

4 Geometric representation of the Hopf cyclic Chern classes

In order to exhibit concrete cocycles representing a basis of $HP^*(\mathcal{K}_n; {}^{\sigma}\mathbb{C})$, we take the same approach as in [19, 20]. The gist of that construction is summarized below.
With $M = \mathbb{R}^n$ and $\mathbf{G} = \mathrm{Diff}(\mathbb{R}^n)^\delta$, let $\{\Omega^\bullet(|\bar{\triangle}_{\mathbf{G}} M|), d\}$ be the complex of Dupont's [11] complex of de Rham simplicial compatible forms. We will

actually work with its homogeneous version $\{\Omega^\bullet(|\bar\triangle_{\mathbf{G}}M|), d\}$. The identification between compatible forms $\omega = \{\omega_p\}_{p\geq 0}$ in the first complex and their homogeneous counterpart $\bar\omega = \{\omega_p\}_{p\geq 0}$ in the second, *i.e.* satisfying

$$\bar\omega(\mathbf{t}; \rho_0\rho, \ldots, \rho_p\rho, \cdot) = \rho^*\bar\omega(\mathbf{t}; \rho_0, \ldots, \rho_p, \cdot), \quad \forall\, \rho, \rho_i \in \mathbf{G},$$

is made via the exchange relations

$$\begin{aligned} &\omega(\mathbf{t}; \phi_1, \ldots, \phi_p, x) = \bar\omega(\phi_1 \cdots \phi_p,\ \phi_2 \cdots \phi_p,\ \ldots, \phi_p, x)\,, \\ \text{resp.}\quad &\bar\omega(\mathbf{t}; \rho_0, \ldots, \rho_p, \cdot) = \rho_p^*\omega(\mathbf{t}; \rho_0\rho_1^{-1}, \rho_1\rho_2^{-1}, \ldots, \rho_{p-1}\rho_p^{-1}, \cdot). \end{aligned}$$

In [20] we introduced the subcomplex $\{\Omega^\bullet_{\rm rd}(|\bar\triangle_{\mathbf{G}}M|), d\}$ of the above complex consisting of *regular differentiable* simplicial de Rham forms. These are compatible forms $\omega = \{\omega_p\}_{p\geq 0}$ on the geometric realization $|\triangle_{\mathbf{G}}M| = \coprod_{p=0}^\infty \Delta^p \times \triangle_{\mathbf{G}}M[p]$ of the simplicial manifold $\triangle_{\mathbf{G}}M = \{\mathbf{G}^p \times M\}_{p\geq 0}$, whose components (expressed in homogeneous group coordinates, but with the "overline" mark omitted from the notation from now on) have the property that

$$\omega_p(\mathbf{t}; \rho_0, \ldots, \rho_p, x) = \sum P_{I,J}\left(\mathbf{t}; x, j^k_x(\rho_0), \ldots, j^k_x(\rho_p)\right) dt_I \wedge dx_J, \qquad (4.1)$$

with $P_{I,J}$ depending polynomially of a finite number of jet components of ρ_a, $1 \leq a \leq p$ and of $\left(\det \rho'_a(x)\right)^{-1}$, where $\rho'_a(x)$ signifies the Jacobian matrix $\rho'_a(x)^j_i = \partial_i \rho^j_a(x)$, $1 \leq i, j \leq n$.
$\Omega^\bullet_{\rm rd}(|\triangle_{\mathbf{G}}M|)$ is a differential graded algebra, whose corresponding cohomology ring $H^\bullet_{\rm rd}(|\bar\triangle_{\mathbf{G}}M|, \mathbb{C})$ was shown in [20, Thm. 1.4] to be isomorphic to the truncated polynomial ring of Chern classes $P_{2n}[c_1, \ldots, c_n]$.
More precisely, let ∇ be the flat connection on the frame bundle $FM \to M$, with connection form $\omega_\nabla = \left(\omega^i_j\right)$, $\omega^i_j := (\mathbf{y}^{-1})^i_\lambda\, d\mathbf{y}^\lambda_j = \left(\mathbf{y}^{-1}\, d\mathbf{y}\right)^i_j$, $i, j = 1, \ldots, n$. The associated simplicial connection form-valued matrix $\hat\omega_\nabla = \{\hat\omega_p\}_{p\in\mathbb{N}}$ on the frame bundle of $|\triangle_{\mathbf{G}}M|$ has components

$$\hat\omega_p(\mathbf{t}; \rho_0, \ldots, \rho_p) := \sum_{i=0}^p t_i \rho^*_i(\omega_\nabla); \qquad (4.2)$$

accordingly, the simplicial curvature form $\hat\Omega_\nabla := d\hat\omega_\nabla + \hat\omega_\nabla \wedge \hat\omega_\nabla$ has compo-

nents

$$
\begin{aligned}
\hat{\Omega}_p(\mathbf{t};\rho_0,\ldots,\rho_p) = & \sum_{i=0}^{p} dt_i \wedge \rho_i^*(\omega_\nabla) - \sum_{i=0}^{p} t_i\rho_i^*(\omega_\nabla) \wedge \rho_i^*(\omega_\nabla) \\
& + \sum_{i,j=0}^{p} t_i t_j\, \rho_i^*(\omega_\nabla) \wedge \rho_j^*(\omega_\nabla).
\end{aligned}
\tag{4.3}
$$

Under the action of $\rho \in \mathbf{G}$ on FM the pull-back of the connection form is

$$
\rho^*(\omega^i_j) = \omega^i_j + \gamma^i_{jk}(\rho)\,\theta^k, \qquad \text{where} \qquad \theta^k = (\mathbf{y}^{-1}\cdot dx)^k \tag{4.4}
$$

$$
\gamma^i_{j\,k}(\rho)(x,\mathbf{y}) = \left(\mathbf{y}^{-1}\cdot\rho'(x)^{-1}\cdot\partial_\lambda\rho'(x)\cdot\mathbf{y}\right)^i_j\, \mathbf{y}^\lambda_k, \quad x\in M,\, \mathbf{y}\in \mathrm{GL}_{\mathbf{n}}(\mathbb{R}).
$$

This clearly shows that the simplicial forms $\hat{\omega}^i_j$ and $\hat{\Omega}^i_j$ belong to the regular differentiable de Rham complex $\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}}FM|)$.
The cohomology $H^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}}M|)$ of the Dupont complex for M was shown in [20] to be isomorphic to the truncated polynomial ring of Chern classes $P_{2n}[c_1,\ldots,c_n]$. More precisely, by [20, Thms. 1.3 and Eq. (1.13)], the choice of the connection gives rise to a canonical quasi-isomorphism of complexes

$$
\mathcal{C}^{\mathrm{GL}_n}_\nabla : \hat{W}(\mathfrak{gl}_n,\mathrm{GL}_n) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}}M|), \tag{4.5}
$$

which in fact reproduces the classical Chern-Weil construction for the Diff-equivariant case. Indeed, the left hand stands for the subalgebras consisting of the GL_n-basic elements in the quotient $\hat{W}(\mathfrak{gl}_n) = W(\mathfrak{gl}_n)/\mathcal{I}_{2n}$ of the Weil algebra $W(\mathfrak{gl}_n) = \wedge^\bullet\mathfrak{gl}_n^*\otimes S(\mathfrak{gl}_n)$ by the ideal generated by the elements of $S(\mathfrak{gl}_n)$ of degree $> 2n$. The cohomology of $\hat{W}(\mathfrak{gl}_n,\mathrm{GL}_n)$ is well-known to be isomorphic to $P_{2n}[c_1,\ldots,c_n]$, with $c_1,\ldots,c_n$ given by the standard generators of the ring $S(\mathfrak{gl}_n)^{\mathrm{GL}_n}$ of $\mathrm{GL}_n(\mathbb{C})$-invariant polynomials on $\mathfrak{gl}_n(\mathbb{C})$,

$$
c_k(A) = \sum_{1\le i_1<\ldots<i_k\le n}\ \sum_{\lambda\in S_k} (-1)^\lambda A^{i_1}_{\lambda(i_1)}\cdots A^{i_k}_{\lambda(i_k)}, \quad A\in\mathfrak{gl}_n(\mathbb{C}). \tag{4.6}
$$

The corresponding Chern forms $c_k(\hat{\Omega}_\nabla)\in\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}}FM|)$ are GL_n-basic and thus descend to forms in $\Omega^{2k}_{\mathrm{rd}}(|\triangle_{\mathbf{G}}M|)$. As a result, the collection of closed forms

$$
c_J(\hat{\Omega}_\nabla) = c_{j_1}(\hat{\Omega}_\nabla)\wedge\ldots\wedge c_{j_q}(\hat{\Omega}_\nabla)\in\Omega^{2|J|}_{\mathrm{rd}}(|\triangle_{\mathbf{G}}M|, \tag{4.7}
$$

with $J = (j_1 \leq \ldots \leq j_q)$ and $|J| := j_1 + \ldots + j_q \leq n$, give a complete set of representatives for a (linear) basis of the cohomology $H^{\bullet}_{\rm rd}(|\bar{\triangle}_{\mathbf{G}} M|, \mathbb{C})$ of the Rham complex $\{\Omega^{\bullet}_{\rm rd}(|\triangle_{\mathbf{G}} M|), d\}$.

Consider now the subcomplex $\{\bar{C}^{\bullet}_{\rm rd}\left(\mathbf{G}, \Omega^{\bullet}(M)\right), \delta, d\}$ of the (homogeneous version of the) Bott bicomplex (see [1, 2]) $\{\bar{C}^{\bullet}\left(\mathbf{G}, \Omega^{\bullet}(M)\right), \delta, d\}$, formed of *regular differentiable* homogeneous group cochains. By definition (*cf.* [20]), a cochain $\omega \in \bar{C}^{p}_{\rm rd}\left(\mathbf{G}, \Omega^{q}(M)\right)$ if for any local chart $U \subset M$ with coordinates $(x^1, \ldots, x^n)$,

$$\omega(\rho_0, \ldots, \rho_p, x) = \sum P_I\left(x, j^k_x(\rho_0), \ldots, j^k_x(\rho_p)\right) dx_I, \tag{4.8}$$

where the coefficients P_I as in (4.1). By [20, Thm. 1.1], which is a variant of Dupont's [11, Theorem 2.3], the operation of integration along the fiber

$$\oint_{\Delta^{\bullet}} : \Omega^{\bullet}_{\rm rd}(|\bar{\triangle}_{\mathbf{G}} M|) \to \bar{C}^{\bullet}_{\rm rd}\left(\mathbf{G}, \Omega^{*}(M)\right) \tag{4.9}$$

establishes a quasi-isomorphism between the complexes $\{\Omega^{\bullet}_{\rm rd}(|\bar{\triangle}_{\mathbf{G}} M|), d\}$ and $\{\bar{C}^{\rm tot}_{\rm rd}\left(\mathbf{G}, \Omega^{*}(M)\right), \delta \pm d\}$. Thus, the composition of (4.5) and (4.9)

$$\mathcal{D}^{\rm GL_n}_{\nabla} := \oint_{\Delta^{\bullet}} \circ \mathcal{C}^{\rm GL_n}_{\nabla} : \hat{W}(\mathfrak{gl}_n, \mathrm{GL}_n) \to \bar{C}^{\rm tot\,\bullet}_{\rm rd}\left(\mathbf{G}, \Omega^{*}(M)\right) \tag{4.10}$$

is also a quasi-isomorphism. In conclusion the cocycles

$$C_J(\hat{\Omega}_{\nabla}) \,:=\, \oint_{\Delta^{\bullet}} c_J(\hat{\Omega}_{\nabla}), \quad J = (j_1 \leq \ldots \leq j_q),\ |J| \leq n\}. \tag{4.11}$$

represent a basis for the cohomology $H^{\bullet}_{\rm rd, \mathbf{G}}(M, \mathbb{C})$ of $\{\bar{C}^{\rm tot}_{\rm rd}\left(\mathbf{G}, \Omega^{*}(M)\right), \delta \pm d\}$.

On the other hand, in [17] we have introduced Hopf cyclic counterparts of the Bott complexes. In particular, in the case of $\mathcal{F}_{\mathcal{K}}$ the analogue of the homogeneous Bott complex is the anti-symmetrized and coinvariant subcomplex of (3.8),

$$\bar{C}^{\bullet}(\wedge V^*, \wedge \mathcal{F}_{\mathcal{K}}) \,=\, (\wedge^p V^* \otimes \wedge \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}}, \tag{4.12}$$

defined as follows. An element $\sum \alpha \otimes \tilde{f} \in (\wedge^p V^* \otimes \wedge^{q+1} \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}}$ if it satisfies the $\mathcal{F}_{\mathcal{K}}$-coinvariance condition:

$$\sum \alpha_{<0>} \otimes\ \tilde{f} \otimes S(\alpha_{<1>}) = \sum\ \alpha \otimes \tilde{f}_{<0>} \otimes \tilde{f}_{<1>}; \tag{4.13}$$

here for $\tilde{f} = f^0 \wedge \cdots \wedge f^q$, we have denoted

$$\tilde{f}_{<0>} \otimes \tilde{f}_{<1>} \;=\; f^0{}_{(1)} \wedge \cdots \wedge f^q{}_{(1)} \otimes f^0{}_{(2)} \cdots f^q{}_{(2)}. \tag{4.14}$$

One identifies the anti-symmetrized-coinvariant bicomplex as a homotopy retraction sub-bicomplex of (3.8) as in [17].

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\wedge^2 V^* \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (\wedge^2 V^* \otimes \wedge^2 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (\wedge^2 V^* \otimes \wedge^3 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & \cdots \\
V^* \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (V^* \otimes \wedge^2 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (V^* \otimes \wedge^3 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & \cdots \\
\mathbb{C} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (\mathbb{C} \otimes \wedge^2 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & (\mathbb{C} \otimes \wedge^3 \mathcal{F}_{\mathcal{K}})^{\mathcal{F}_{\mathcal{K}}} \arrow[u, "\partial_\wedge"] \arrow[r, "b_\wedge"] & \cdots
\end{tikzcd}, \tag{4.15}$$

The identification simplifies the action of V on $\bigwedge \mathcal{F}_{\mathcal{K}}$ into the diagonal action

$$X \triangleright (f^0 \otimes \cdots \otimes f^q) = \sum_{i=0}^{q} f^0 \otimes \cdots \otimes X \triangleright f^i \otimes \cdots \otimes f^q. \tag{4.16}$$

and also the coboundaries are simplified to

$$b_\wedge(\alpha \otimes f^0 \wedge \cdots \wedge f^q) = \alpha \otimes 1 \wedge f^0 \wedge \cdots \wedge f^q, \tag{4.17}$$

and

$$\partial_\wedge(\alpha \otimes f^0 \wedge \cdots \wedge f^q) = -\sum_i \theta^i \wedge \alpha \otimes \wedge \otimes X_i \triangleright (f^0 \wedge \cdots \wedge f^q). \tag{4.18}$$

The total cohomology of this bicomplex is denoted by $HP^\bullet_{\rm CE}(\mathcal{K}_n, {}^{\sigma^{-1}}\mathbb{C})$
A similar bicomplex is defined for $\mathcal{F}_{\mathcal{H}}$),

$$\bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{H}}) := (\wedge^p V^* \otimes \wedge^{q+1} \mathcal{F}_{\mathcal{H}})^{\mathcal{F}_{\mathcal{H}}}, \tag{4.19}$$

and the restriction map $r_{\mathcal{H}} : \mathcal{F}_{\mathcal{K}} \to \mathcal{F}_{\mathcal{H}}$ induces a quasi-isomorphism between (4.15) and (4.19).

The total cohomology of this bicomplex is denoted by $HP^{\bullet}_{\mathrm{CE}}(\mathcal{H}_n, \mathrm{GL}_n : \mathbb{C}_\delta)$
At this stage we recall that in [17, §3.2] we have constructed a map of bicomplexes, Θ from the bicomplex $\bar{C}^{\bullet}(\wedge\mathfrak{g}^*_{\mathrm{aff}}, \wedge\mathcal{F}_{\mathcal{H}})$ of antisymmetrized $\mathcal{F}_{\mathcal{H}}$-coinvariant cochains to $\bar{C}^{\bullet}(\mathbf{G}, \Omega^*(FM))$. In order to write its expression, we also need to recall (*cf.* [16]) that there is a canonical isomorphism $\not\eta : \mathcal{H}^{\mathrm{cop}}_{\mathrm{ab}} \to \mathcal{F}_{\mathcal{H}}$ and $\not\delta = \not\eta^{-1}$ denotes its inverse. For $f \in \mathcal{F}_{\mathcal{H}}$, one defines the function $\not\gamma_{\mathcal{H}}(f) : \mathbf{G} \to C^\infty(G)$ by

$$\not\delta(S(f))(U_\phi) \;=\; \not\gamma_{\mathcal{H}}(f)(\phi)\, U^*_\phi, \qquad \forall\, \phi \in \mathbf{G}, \tag{4.20}$$

where the left hand side uses the action of $\mathcal{H}_n$ on the crossed product algebra $C^\infty(G_{\mathrm{aff}}) \rtimes \mathbf{G}$, with $G_{\mathrm{aff}} = V \ltimes \mathrm{GL}_n(\mathbb{R})$. The function $\not\gamma_{\mathcal{H}}(f)(\phi) \in C^\infty(G)$, depends algebraically on the components of the k-jet of ϕ, for some $k \in \mathbb{N}$. For example, if $f = \eta^i_{jk\ell_1\ldots\ell_r}$ is one of the canonical algebra generators of $\mathcal{F}_{\mathcal{H}}$,

$$\eta^i_{jk\ell_1\ldots\ell_r}(\psi) = \gamma^i_{jk\ell_1\ldots\ell_r}(\psi)(e), \quad \psi \in \mathbf{N} \tag{4.21}$$

then

$$\not\gamma_{\mathcal{H}}(S(\eta^i_{jk\ell_1\ldots\ell_r}))(\psi^{-1}) \;=\; \gamma^i_{jk\ell_1\ldots\ell_r}(\psi). \tag{4.22}$$

With these notions clarified, the map Θ is given by the formula

$$\begin{aligned} &\Theta\big(\sum_I \alpha_I \otimes {}^I\!f^0 \wedge \cdots \wedge {}^I\!f^p\big)(\phi_0, \ldots, \phi_p) = \\ &\sum_I \sum_{\sigma \in S_{p+1}} (-1)^\sigma \not\gamma_{\mathcal{H}}(S({}^I\!f^{\sigma(0)}))(\phi_0^{-1}) \ldots \not\gamma_{\mathcal{H}}(S({}^I\!f^{\sigma(p)}))(\phi_p^{-1})\tilde{\alpha}_I, \end{aligned} \tag{4.23}$$

where $\mathfrak{g}_{\mathrm{aff}}$ is the Lie algebra of the affine group $G_{\mathrm{aff}} \cong FM$, and $\{\tilde{\alpha}_I\}$ are the left-invariant form associated to the elements of a basis $\{\alpha_I\} \subset \wedge^{\bullet}\mathfrak{g}^*_{\mathrm{aff}}$.
From its very definition, it is obvious that Θ actually lands in $\bar{C}^{\bullet}_{\mathrm{rd}}(\mathbf{G}, \Omega^*(FM))$.
It is also transparent that Θ is injective.
On the other hand, Θ is clearly GL_n-equivariant and thus, by restriction to GL_n-invariants, it gives the map $\Theta^{\mathrm{GL}_n} : \bar{C}^{\bullet}(\wedge V^*, \wedge\mathcal{F}_{\mathcal{H}}) \to \bar{C}^{\bullet}_{\mathrm{rd}}(\mathbf{G}, \Omega^*(M))$,

$$\begin{aligned} &\Theta^{\mathrm{GL}_n}\big(\sum_{|I|=q} dx_I \otimes {}^I\!f^0 \wedge \cdots \wedge {}^I\!f^p\big)(\phi_0, \ldots, \phi_p) = \\ &\sum_I \sum_{\sigma \in S_{p+1}} (-1)^\sigma \not\gamma_{\mathcal{H}}(S({}^I\!f^{\sigma(0)}))(\phi_0^{-1}) \ldots \not\gamma_{\mathcal{H}}(S({}^I\!f^{\sigma(p)}))(\phi_p^{-1}) dx_I. \end{aligned} \tag{4.24}$$

Theorem 4.1. *The chain map* $\Theta^{\mathrm{GL}_n} : \bar{C}^{\bullet}(\wedge V^*, \wedge\mathcal{F}_{\mathcal{H}}) \to \bar{C}^{\bullet}_{\mathrm{rd}}(\mathbf{G}, \Omega^*(M))$ *is a quasi-isomorphism.*

Proof. The proof follows along exactly the same lines as that of [19, Thm. 3.6], the only difference being that all the *differentiable subcomplexes* are replaced by their *regular differentiable* counterparts. This also entails replacing the horizontal homotopy used in the proof of [19, Thm. 1.2] by the algebraic homotopy employed in the proof of [20, Thm. 1.3]. □

The preimage by Θ^{GL_n} of the cocycles $C_J(\hat{\Omega}_\nabla)$ defined by (4.11) can be exactly computed. Indeed, first we observe that by [17, Remark 3.9] the map Θ is insensitive to affine transformations, *i.e.* if $\varphi_0, \ldots, \varphi_q \in G_{\mathrm{aff}}$ and $\psi_0, \ldots, \psi_q \in \mathbf{N}$, then

$$\begin{aligned}\Theta\big(\sum_I \alpha_I \otimes\ {}^If_0 \wedge \cdots \wedge\ {}^If_q\big)(\varphi_0\psi_0, \ldots, \varphi_q\psi_q) \ =& \\ \Theta\big(\sum_I \alpha_I \otimes\ {}^If_0 \wedge \cdots \wedge\ {}^If_q\big)(\psi_0, \ldots, \psi_q).&\end{aligned} \tag{4.25}$$

Next we note that being given by invariant polynomials, the Chern cocycles (4.11) are built out of the pull-back of the curvature form by the cross-section $x \in \mathbb{R}^n \mapsto (x, \mathbf{1}) \in \mathbb{R}^n \times \mathrm{GL}_n$. The resulting simplicial matrix-valued form is

$$\begin{aligned}\hat{R}(\mathbf{t}; \phi_0, \ldots, \phi_p) \ =& \sum_{r=0}^{p} dt_r \wedge \Gamma(\phi_r) - \sum_{r=0}^{p} t_r\, \Gamma(\phi_r) \wedge \Gamma(\phi_r) \\ &+ \sum_{r,s=0}^{p} t_r t_s\, \Gamma(\phi_r) \wedge \Gamma(\phi_s), \qquad \text{where} \quad \Gamma(\phi) := (\phi')^{-1} \cdot d\phi',\end{aligned} \tag{4.26}$$

which by restriction to $|\bar{\triangle}_{\mathbf{N}} M|$ becomes

$$\hat{R}(\mathbf{t}; \psi_0, \ldots, \psi_p) = \sum_{r=0}^{p} dt_r \wedge d\psi'_r - \sum_{r=0}^{p} t_r\, d\psi'_r \wedge d\psi'_r + \sum_{r,s=0}^{p} t_r t_s\, d\psi'_r \wedge d\psi'_s. \tag{4.27}$$

For $\psi \in \mathbf{N}$, $(d\psi')^i_j \ = \ \sum_{k=1}^n \partial_k \partial_j \psi^i \, dx^k$ and so $(d\psi')^i_j \,|_{x=0} = \ \sum_{k=1}^n \eta^i_{jk}(\psi)\, dx^k$. This clearly shows that the restriction of the simplicial Chern form $c_J(\hat{\Omega}_\nabla)$ to

$|\bar{\triangle}_{\mathbf{N}} M|$ evaluated at $x=0$ gives by integration over the simplices a cocycle $C_J(\hat{R}|_0) \in C^\bullet_{\mathcal{F}}(\wedge M^*, \wedge \mathcal{F}_{\mathcal{H}})$. Moreover, by the very construction,

$$\Theta^{\mathrm{GL}_n}\big(C_J(\hat{R}|_0)\big) \,=\, C_J(\hat{\Omega}_\nabla). \tag{4.28}$$

Combining Theorem 4.1 with the statement (4.11) we obtain:

Corollary 4.2. *The cocycles $C_J(\hat{R}|_0) \in \bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{H}})$, with $J = (j_1 \leq \ldots \leq j_q)$, $|J| \leq n$, represent classes which form a basis for the cohomology $HP^\bullet_{\mathrm{CE}}(\mathcal{H}_n, \mathrm{GL}_n : \mathbb{C}_\delta)$.*

To reproduce the same approach for the algebra $\mathcal{K}_n$, we replace the map Θ by its counterpart corresponding to the decomposition $\mathbf{G} = \mathbf{T} \cdot \mathbf{G}^\dagger$. The new chain map $\Theta_{\mathcal{K}}$, from the subcomplex $\bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{K}})$ of $\mathcal{F}_{\mathcal{K}}$-coinvariant cochains in $C^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{K}})$ to $\bar{C}^\bullet(\mathbf{G}, \Omega^*(FM))$, is defined by the similar formula

$$\begin{aligned}\Theta_{\mathcal{K}}\Big(\sum_I \alpha_I \otimes\, {}^I\!f^0 \wedge \cdots \wedge\, {}^I\!f^p\Big)(\phi_0, \ldots, \phi_p) = \\ \sum_I \sum_{\sigma \in S_{p+1}} (-1)^\sigma \curlyvee_{\mathcal{K}}(S(\,{}^I\!f^{\sigma(0)}))(\phi_0^{-1}) \ldots \curlyvee_{\mathcal{K}}(S(\,{}^I\!f^{\sigma(p)}))(\phi_p^{-1}) \tilde{\alpha}_I.\end{aligned} \tag{4.29}$$

The analogous property to (4.25) reads as follows: if $\varphi_0, \ldots, \varphi_q \in \mathbf{T}$ and $\psi_0, \ldots, \psi_q \in \mathbf{G}^\dagger$, then

$$\begin{aligned}\Theta_{\mathcal{K}}\Big(\sum_I \alpha_I \otimes\, {}^I\!f_0 \wedge \cdots \wedge\, {}^I\!f_q\Big)(\varphi_0\psi_0, \ldots, \varphi_q\psi_q) \,= \\ \Theta_{\mathcal{K}}\Big(\sum_I \alpha_I \otimes\, {}^I\!f_0 \wedge \cdots \wedge\, {}^I\!f_q\Big)(\psi_0, \ldots, \psi_q).\end{aligned} \tag{4.30}$$

This follows from the simple fact that for any translation $\varphi(x) = x + b$, one has $\sigma^i_j(U^*_\varphi) = \delta^i_j U^*_\varphi$.
The dual to the projection map $\psi \in \mathbf{G}^\dagger \mapsto \nu_\psi \in \mathbf{N}$ gives an inclusion $\iota_{\mathcal{H}} : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{K}}$, which is defined by

$$\iota_{\mathcal{H}}(f) = \Phi^{-1}(1 \otimes f),$$

where Φ^{-1} is defined in (2.48). One observes that $\iota_{\mathcal{H}}$ is a cross-section of the restriction map $r_{\mathcal{H}} : \mathcal{F}_{\mathcal{K}} \to \mathcal{F}_{\mathcal{H}}$. In turn, $\iota_{\mathcal{H}}$ gives rise to a chain map $\iota^\bullet_{\mathcal{H}} : \bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{H}}) \to \bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{K}})$ at the level of Hochschild complexes. Manifestly, one has

$$\Theta_{\mathcal{K}} \circ \iota^\bullet_{\mathcal{H}} \,=\, \Theta^{\mathrm{GL}_n}. \tag{4.31}$$

Lemma 4.3. *The chain map* $\iota^{\bullet}_{\mathcal{H}} : \bar{C}^{\bullet}(\wedge V^*, \wedge\mathcal{F}_{\mathcal{H}}) \to \bar{C}^{\bullet}(\wedge V^*, \wedge\mathcal{F}_{\mathcal{K}})$ *is a quasi-isomorphism of bicomplexes.*

Proof. By construction, $r^{\bullet}_{\mathcal{H}} \circ \iota^{\bullet}_{\mathcal{H}} = \mathrm{Id}$. On the other hand the very same arguments invoked in the proof of Theorem 3.3 show that $r^{\bullet}_{\mathcal{H}}$ is a quasi-isomorphism in Hochschild cohomology, which moreover induces an isomorphism in cyclic cohomology. Therefore, its right inverse $\iota^{\bullet}_{\mathcal{H}}$, which is a chain map of bicomplexes, gives an isomorphism in the cohomology of the total complexes. ☐

We next build the preimage by $\Theta_{\mathcal{K}}$ of the cocycles $C_J(\hat{\Omega}_{\nabla})$ in exactly the same fashion as for Θ^{GL_n}, except that instead of using the simplicial curvature (4.27) on $|\Delta_{\mathbf{N}}M|$, we use the simplicial curvature form (4.26) on $|\Delta_{\mathbf{G}^{\dagger}}M|$. Note that the latter involves the forms $\Gamma(\psi) := (\psi')^{-1}\cdot d\psi'$, for $\psi \in \mathbf{G}^{\dagger}$. The cocycles thus obtained, $C^{\dagger}_J(\hat{R}|_0)$ are uniquely determined by the equation

$$\Theta_{\mathcal{K}}\big(C^{\dagger}_J(\hat{R}|_0)\big) \;=\; C_J(\hat{\Omega}_{\nabla}). \tag{4.32}$$

Corollary 4.4. *The cocycles* $C^{\dagger}_J(\hat{R}|_0) \in \bar{C}^{\bullet}(\wedge V^*, \wedge\mathcal{F}_{\mathcal{K}})$*, with* $J = (j_1 \leq \ldots \leq j_q)$*,* $|J| \leq n$*, represent a basis of cohomology classes for* $HP^{\bullet}_{\mathrm{CE}}(\mathcal{K}_n; {}^{\sigma^{-1}}\mathbb{C})$*.*

Proof. The relation (4.31) implies that $\iota_{\mathcal{H}}(C_J(\hat{R}|_0)) = C^{\dagger}_J(\hat{R}|_0)$, since $\Theta_{\mathcal{K}}$ is injective. The claim then follows from Lemma 4.3. ☐

5 Characteristic map and Hopf cyclic Chern cocycles

The crossed product algebra $\mathcal{A} = C^{\infty}_c(M) \rtimes \mathbf{G}$ has a canonical state-like functional $\tau : \mathcal{A} \to \mathbb{C}$, determined by the standard volume form on $M = \mathbb{R}^n$, $\varpi = dx^1 \wedge \ldots \wedge dx^n$; it is given by

$$\tau(fU^*_{\phi}) = \begin{cases} \displaystyle\int_M f\varpi, & \text{if} \quad \phi = \mathrm{Id} \\ \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

Unlike its forerunner on the frame bundle employed in [7, 8], the linear map τ is not a trace. It is however easy to check that τ is a σ^{-1}-trace, *i.e.*

$$\tau(ab) = \tau(b\sigma^{-1}(a)), \qquad \forall\, a, b \in \mathcal{A}, \tag{5.2}$$

and that it is ε-invariant with respect to the action of $\mathcal{K}_n$, meaning that

$$\tau(k(a)) = \varepsilon(k)\tau(a), \qquad \forall\, k \in \mathcal{K}_n,\, a \in \mathcal{A}; \tag{5.3}$$

in particular, $\tau(\sigma(a)) = \tau(a)$.
Having these two properties, one can define (*cf.* [7, 9]) a characteristic map χ_τ from the standard Hopf cyclic (b,B)-complex $CC^\bullet(\mathcal{K}_n;\, {}^{\sigma^{-1}}\mathbb{C})$ to the cyclic cohomology (b,B)-complex $CC^\bullet(\mathcal{A})$, by

$$\begin{gathered}\chi_\tau(k^1,\ldots,k^q)(a_0\ldots,a_q) = \tau(a_0k^1(a^1)\cdots k^q(a_q)),\\ k^1,\ldots,k^q \in \mathcal{K}_n, \qquad a_0,\ldots,a_q \in \mathcal{A}.\end{gathered}$$

which is a map of cyclic complexes. As a matter of fact, this map is injective and the model for the Hopf cyclic structure in the left hand side was originally imported in [7] from that of the right hand side. We will show below that this structural characteristic map also allows to transfer the geometric cocycles constructed in §4 to the Hopf cyclic complex $CC^\bullet(\mathcal{K}_n;\, {}^{\sigma^{-1}}\mathbb{C})$.

Connes has constructed (see [5, III.2.δ]) a map of bicomplexes

$$\Phi_{\mathrm{C}} : \bar{C}^\bullet(\Gamma, \Omega^\bullet(M)) \to CC^\bullet(C_c^\infty(M) \rtimes \mathbf{G}),$$

whose definition we quickly recall.
Let $\mathcal{B}_{\mathbf{G}}(M)$ denote the DG-algebra $\Omega_c^*(M) \otimes \wedge\mathbb{C}[\mathbf{G}']$, where $\mathbf{G}' = \mathbf{G} \setminus \{e\}$ with the differential $d \otimes \mathrm{Id}$. After labeling the generators of $\mathbb{C}[\mathbf{G}']$ as γ_ϕ, $\phi \in \mathbf{G}$, with $\gamma_1 = 0$, one forms the crossed product $\mathcal{C}_{\mathbf{G}}(M) = \mathcal{B}_{\mathbf{G}}(M) \rtimes \mathbf{G}$, with the multiplication rules

$$\begin{aligned} &U_\phi^*\, \omega\, U_\phi = \phi^*\, \omega, && \omega \in \Omega_c^*(M),\\ &U_{\phi_1}^*\, \gamma_{\phi_2}\, U_{\phi_1} = \gamma_{\phi_2\circ\phi_1} - \gamma_{\phi_1}, && \phi_1,\phi_2 \in \mathbf{G}\,. \end{aligned}$$

$\mathcal{C}_{\mathbf{G}}(M)$ is itself a DG-algebra, equipped with the differential

$$d(b\, U_\phi^*) = db\, U_\phi^* - (-1)^{\partial b}\, b\, \gamma_\phi\, U_\phi^*, \qquad b \in \mathcal{B}_{\mathbf{G}}(G), \quad \phi \in \mathbf{G}, \tag{5.4}$$

Any $\lambda \in \bar{C}^q(\mathbf{G}, \Omega^p(M))$ gives rise a linear form $\widetilde{\lambda}$ on $\mathcal{C}_{\mathbf{G}}(G)$ as follows:

$$\begin{aligned} &\widetilde{\lambda}(b\, U_\phi^*) = 0 \qquad \text{for} \quad \phi \neq 1;\\ &\text{if} \quad \phi = 1 \quad \text{and} \quad b = \omega \otimes \gamma_{\rho_1}\ldots\gamma_{\rho_q} \qquad \text{then}\\ &\widetilde{\lambda}(\omega \otimes \gamma_{\rho_1}\ldots\gamma_{\rho_q}) = \int_M \lambda(1,\rho_1,\ldots,\rho_q) \wedge \omega. \end{aligned} \tag{5.5}$$

The map $\Phi_{\rm C}$ from $\bar{C}^\bullet(\mathbf{G}, \Omega^\bullet(G))$ to the (b, B)-complex of the algebra $\mathcal{A} = C_c^\infty(M) \rtimes \mathbf{G}$ is now defined for $\lambda \in \bar{C}^q(\mathbf{G}, \Omega^p(M))$ by

$$\Phi_{\rm C}(\lambda)(a^0, \ldots, a^m) = \frac{p!}{(m+1)!} \sum_{j=0}^{m} (-1)^{j(m-j)} \widetilde{\lambda}(da^{j+1} \cdots da^m \, a^0 \, da^1 \cdots da^j)$$
$$\text{where} \quad m = \dim G - p + q, \qquad a^0, \ldots, a^m \in \mathcal{A}. \tag{5.6}$$

As proved in [5, III.2, Thm. 14], $\Phi_{\rm C}$ is a chain map to the total (b, B)-complex of the algebra $\mathcal{A}$.
We denote by $\Phi_{\rm rd}$ the restriction of $\Phi_{\rm C}$ to the subcomplex

$$\bar{C}^{\rm tot}_{\Theta}(\mathbf{G}, \Omega^*(M)) := \Theta_{\mathcal{K}}\big(\bar{C}^\bullet(\wedge V^*, \wedge \mathcal{F}_{\mathcal{K}})\big) \subset \bar{C}^{\rm tot}_{\rm rd}(\mathbf{G}, \Omega^*(M)).$$

By reasoning as in [7, pp 223-234], it can be shown that if $\lambda \in \bar{C}^q_{\Theta}(\mathbf{G}, \Omega^p(M))$ then there exists $\tilde{k}(\lambda) = \sum_\alpha \dot{k}^1_\alpha \otimes \ldots \otimes \dot{k}^q_\alpha \in \mathcal{K}_n^{\otimes q}$ such that

$$\Phi_{\rm C}(\lambda) \,=\, \sum_\alpha \chi_\tau(k^1_\alpha, \ldots, k^q_\alpha);$$

due to the faithfulness of χ_τ, the element $\tilde{k}(\lambda)$ is necessarily unique. This gives a canonical identification between the two (b, B)-complexes,

$$CC^\bullet(\mathcal{K}_n; {}^{\sigma^{-1}}\mathbb{C}) \cong {\rm Im}(\Phi_{\rm rd}), \tag{5.7}$$

which allows us to regard $\Phi_{\rm rd}$ as a chain map to $CC^{\rm tot}(\mathcal{K}_n \, {}^{\sigma^{-1}}\mathbb{C})$.

Theorem 5.1. *The map $\Phi_{\rm rd} : \bar{C}^{\rm tot}_{\Theta}(\mathbf{G}, \Omega^*(M)) \to CC^{\rm tot}(\mathcal{K}_n; {}^{\sigma^{-1}}\mathbb{C})$ is a quasi-isomorphism. Moreover, via the above identification, the cocycles*

$$\kappa_J(\hat{\Omega}_\nabla) := \Phi_{\rm rd}\big(C_J(\hat{\Omega}_\nabla)\big) \in CC^{\rm tot}(\mathcal{K}_n \, {}^{\sigma^{-1}}\mathbb{C}), \tag{5.8}$$

with $J = (j_1 \leq \ldots \leq j_q)$, $|J| \leq n$, represent a basis of cohomology classes for $HP^\bullet(\mathcal{K}_n; {}^{\sigma^{-1}}\mathbb{C})$.

Proof. By construction,

$$\Phi_{\rm rd} \circ \Theta_{\mathcal{K}} \circ \iota^\bullet_{\mathcal{H}} = \Phi_{\rm rd} \circ \Theta^{{\rm GL}_n}.$$

The right hand side was shown to be a quasi-isomorphism in [20, §2.2], while $\iota^\bullet_{\mathcal{H}}$ is quasi-isomorphism by Lemma 4.3. ∎

6 Explicit calculations for $\mathcal{K}_1$

We illustrate the above results, by producing explicit cocycles for the Hopf cyclic classes of $\mathcal{K}_1$.
The connection form on $FM = \mathbb{R} \times \mathbb{R}^\times$ being $\omega \equiv \omega_1^1 := \mathbf{y}^{-1}\, d\mathbf{y}$, the associated simplicial connection form is

$$\hat{\omega}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) := \sum_{i=0}^{p} t_i \rho_i^*(\omega) = \sum_{i=1}^{p} s_i(\rho_{i-1}^*(\omega) - \rho_i^*(\omega)) + \rho_p^*(\omega).$$

The pull-back of the connection form is

$$\rho^*(\omega_1^1) = \omega_1^1 + \gamma_{11}^1(\rho)\, \mathbf{y}^{-1} \cdot dx,$$
$$\gamma_{1\,1}^1(\rho)(x, \mathbf{y}) = \mathbf{y}^{-1} \cdot \rho'(x)^{-1} \cdot \partial\rho'(x) \cdot \mathbf{y}\, \mathbf{y},$$

that is

$$\rho^*(\omega) \;=\; \mathbf{y}^{-1} d\mathbf{y} \;+\; \frac{\rho''}{\rho'} dx.$$

One has

$$\hat{\omega}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) = \sum_{i=1}^{p} s_i \left(\frac{\rho''_{i-1}}{\rho'_{i-1}} - \frac{\rho''_i}{\rho'_i} \right) dx + \frac{\rho''_p}{\rho'_p} dx + \mathbf{y}^{-1} d\mathbf{y}.$$

The simplicial curvature form $\hat{\Omega} = d\hat{\omega}_\nabla + \hat{\omega}_\nabla \wedge \hat{\omega}_\nabla$ has components

$$\begin{aligned}\hat{\Omega}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) = &\sum_{i=1}^{p} \left(\frac{\rho''_{i-1}}{\rho'_{i-1}} - \frac{\rho''_i}{\rho'_i} \right) ds_i dx \\ &+ \left(\sum_{i=1}^{p} s_i \left(\frac{\rho''_{i-1}}{\rho'_{i-1}} - \frac{\rho''_i}{\rho'_i} \right) + \frac{\rho''_p}{\rho'_p} \right) \mathbf{y}^{-1} dx d\mathbf{y}.\end{aligned}$$

Its pull-back by the canonical section is the curvature form

$$\hat{R}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) = \sum_{i=1}^{p} \left(\frac{\rho''_{i-1}}{\rho'_{i-1}} - \frac{\rho''_i}{\rho'_i} \right) ds_i dx.$$

The image $\{\oint_{\Delta_p} \hat{R}_p\}_p$ in the Bott complex is nontrivial only for $p = 1$, giving the cochain $c_1 \in \bar{C}^1(\mathbf{G}, \Omega^1(M))$,

$$c_1(\rho_0, \rho_1) = \left(\frac{\rho''_0}{\rho'_0} - \frac{\rho''_1}{\rho'_1} \right) dx \tag{6.1}$$

Let us compute $\Phi_{\mathrm{C}}(\tilde{c}_1)$. Recall that $\tilde{c}_1$ is the current

$$\tilde{c}_1(\alpha \otimes \gamma_\varphi) = \int_M c_1(1,\varphi) \wedge \alpha, \quad a \in \Omega_c^q(M), \tag{6.2}$$

which only pairs nontrivially if $\alpha \in \Omega_c^0(M)$. Then

$$\Phi_{\mathrm{C}}(c_1)(a_0, a_1) = \frac{1}{2}\tilde{c}_1(da_1\, a_0 + a_0\, da_1), \quad a_0, a_1 \in \mathcal{A}. \tag{6.3}$$

Take $a_0 = f_0 U^*_{\rho_0}$ and $a_1 = f_1 U^*_{\rho_1}$ with $\rho_1\rho_0 = 1$. Then $a := a_0\, a_1 = f_0\, \rho^*(f_1) := f$, hence

$$\tilde{c}_1(da) = \tilde{c}_1(df) = 0. \tag{6.4}$$

So we can rewrite (6.3) as

$$\Phi_{\mathrm{C}}(c_1)(a_0, a_1) = \frac{1}{2}\tilde{c}_1(da_1\, a_0 - da_0\, a_1), \quad a_0, a_1 \in \mathcal{A}. \tag{6.5}$$

One has

$$\begin{aligned} da_1\, a_0 - da_0\, a_1 &= (df_1 U^*_{\rho_1} - f_1 \gamma_{\rho_1} U^*_{\rho_1}) f_0 U^*_{\rho_0} - (df_0 U^*_{\rho_0} - f_0 \gamma_{\rho_0} U^*_{\rho_0}) f_1 U^*_{\rho_1} \\ &= df_1 U^*_{\rho_1} f_0 U^*_{\rho_0} - f_1 \gamma_{\rho_1} U^*_{\rho_1} f_0 U^*_{\rho_0} - df_0 U^*_{\rho_0} f_1 U^*_{\rho_1} + f_0 \gamma_{\rho_0} U^*_{\rho_0} f_1 U^*_{\rho_1} \\ &= df_1\, {\rho_1}^*(f_0) - f_1\, {\rho_1}^*(f_0)\, \gamma_{\rho_1} - df_0\, {\rho_0}^*(f_1) + f_0\, {\rho_0}^*(f_1)\, \gamma_{\rho_0}. \end{aligned}$$

Hence

$$\begin{aligned} 2\Phi_{\mathrm{C}}(c_1)(a_0, a_1) &= \tilde{c}_1(f_0\, {\rho_0}^*(f_1)\, \gamma_{\rho_0} - f_1\, {\rho_1}^*(f_0)\, \gamma_{\rho_1}) \\ &= \int_M f_0\, {\rho_0}^*(f_1) c_1(1, \rho_0) - \int_M f_1\, {\rho_1}^*(f_0) c_1(1, \rho_1) \end{aligned}$$

So letting $\rho_0 = \phi, \rho_1 = \phi^{-1}$, one has

$$\begin{aligned} 2\Phi_{\mathrm{C}}(c_1)(a_0, a_1) &= \int_M f_0\, {\rho_0}^*(f_1) \left(-\frac{\rho_0''}{\rho_0'}\right) dx - \int_M f_1\, {\rho_1}^*(f_0) \left(-\frac{\rho_1''}{\rho_1'}\right) dx \\ &= -\int_M f_0\, (f_1 \circ \phi)\, \frac{\phi''}{\phi'}\, dx + \int_M f_1\, (f_0 \circ \phi^{-1})\, \frac{(\phi^{-1})''}{(\phi^{-1})'}\, dx. \end{aligned}$$

Note now that, by substitution one has

$$\int_M f_0(x)\,(f_1(\phi(x))\,\frac{\phi''(x)}{\phi'(x)}\,dx = \int_M f_1(x)\,(f_0(\phi^{-1}(x))\frac{\phi''(\phi^{-1}(x)}{\phi'(\phi^{-1}(x)}\,(\phi^{-1})'(x)dx$$
$$= \int_M f_1(x)\,(f_0(\phi^{-1}(x))\phi''(\phi^{-1}(x)\,(\phi^{-1})'(x)^2dx$$
$$= -\int_M f_1(x)\,(f_0(\phi^{-1}(x))\,\frac{(\phi^{-1})''(x)}{(\phi^{-1})'(x))}dx;$$

the last equality uses the elementary identity

$$\phi''(\phi^{-1}(x))\,(\phi^{-1})'(x)^2 \,+\, \frac{(\phi^{-1})''(x)}{(\phi^{-1})'(x))} \,=\, 0$$

Thus we get

$$\begin{aligned} 2\Phi_{\mathrm{C}}(c_1)(a_0,a_1) &= 2\int_M f_1\,(f_0\circ\phi^{-1})\,\frac{(\phi^{-1})''}{(\phi^{-1})'}\,dx \\ &= 2\tau(\sigma^{-1}\sigma^1_{11}(a_1)a_0) \,=\, 2\tau(a_0\,\sigma^{-2}\sigma^1_{11}(a_1)). \end{aligned}$$

Equivalently,

$$\Phi_{\mathrm{C}}(c_1)(a_0,a_1) \,=\, \chi_\tau(\sigma^{-2}\sigma^1_{11})(a_0,a_1). \tag{6.6}$$

The other class arises from the constant simplicial form $\mathbf{1} \in \Omega^0(|\Delta_{\mathbf{G}}M|)$, which gives the cochain $c_0 \in \bar{C}^0(\mathbf{G},\Omega^0(M))$,

$$c_0(\rho) \equiv 1\,; \tag{6.7}$$

thus $\tilde{c}_0$ is the "transverse fundamental" current

$$\tilde{c}(\alpha) \,=\, \int_M \alpha, \qquad \alpha\in\Omega^0(M).$$

As in the previous case, taking $\rho_0 = \phi, \rho_1 = \phi^{-1}$ and using the similar observation $\tilde{c}_0(da) = 0$, one has

$$\begin{aligned} 2\Phi_{\mathrm{C}}(c_0)(a_0,a_1) &= \tilde{c}_0(da_1\,a_0 - da_0\,a_1) = \tilde{c}_0\big(df_1\,{\rho_1}^*(f_0) - df_0\,{\rho_0}^*(f_1)\big) \\ &= \int_M (f_0\circ\phi^{-1})df_1 - \int_M (f_1\circ\phi)df_0 \\ &= \int_M (f_0\circ\phi^{-1})X_1(f_1)dx - \int_M (f_1\circ\phi)X_1(f_0)dx \\ &= 2\int_M (f_0\circ\phi^{-1})X_1(f_1)dx = 2\tau(X_1(a_1)a_0) = 2\tau(a_0\sigma^{-1}X_1(a_1)). \end{aligned}$$

Thus,

$$\Phi_{\mathrm{C}}(c_0)(a_0, a_1) \;=\; \chi_\tau(\sigma^{-1}X_1)(a_0, a_1). \tag{6.8}$$

Summing up, we have established the following result.

Proposition 6.1. *The formulas* (6.8) *and* (6.6) *determine uniquely the cyclic cocycles* C_0 *and* C_1 *in* $CC^1(\mathcal{K}_1, {}^{\sigma^{-1}}\mathbb{C})$,

$$C_0 := \mathbf{1} \otimes \sigma^{-1}X_1, \qquad C_1 := \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1}\,, \tag{6.9}$$

whose cohomology classes form a basis of $HP^\bullet(\mathcal{K}_1, {}^{\sigma^{-1}}\mathbb{C})$.

Concerning the relative characteristic map for $(\mathcal{H}_n, \mathrm{GL}_n)$, in dimensions $n \geq 2$ it should be considered as mapping to the complex $\{CC^\bullet(C_c^\infty(M) \rtimes \mathbf{G}_\omega), b, B\}$, where $\mathbf{G}_\omega = \mathrm{Diff}_\omega(M)$ is the group of volume preserving diffeomorphisms of M. The case $n = 1$ is exceptional because $\mathrm{Diff}_\omega(\mathbb{R}) \cong \mathbb{R}$ reduces to the group of translations. However, the cocycles $c_0, c_1 \in \bar{C}^\bullet(\mathbf{G}, \Omega^\bullet(\mathbb{R}))$ given by (6.7) and (6.1) do determine uniquely cocycles representing a basis for the Hopf cyclic of $\mathcal{H}_1$, but in the model given by the Hopf-Chevalley-Eilenberg bicomplex (4.19); these cocycles are

$$C_0(\hat{R}|_0) = 1 \in \mathcal{F}_{\mathcal{H}}, \quad C_1(\hat{R}|_0) = \theta^1 \otimes 1 \wedge \alpha^1_{1,1} \in V^* \otimes \wedge^2 \mathcal{F}_{\mathcal{H}}. \tag{6.10}$$

Similar representatives are determined in the bicomplex (4.15) giving the cohomology $HP^\bullet_{\mathrm{CE}}(\mathcal{K}_1, {}^{\sigma^{-1}}\mathbb{C})$, via the map $\Theta_{\mathcal{K}}$, namely

$$C_0^\dagger(\hat{R}|_0) = 1 \in \mathcal{F}_{\mathcal{K}}, \quad C_1^\dagger(\hat{R}|_0) = \theta^1 \otimes 1 \wedge \beta^{-1}\beta^1_{1,1} \in V^* \otimes \wedge^2 \mathcal{F}_{\mathcal{K}}. \tag{6.11}$$

In contrast to the case of $\mathcal{H}_1$, the Hopf cyclic cohomology of $\mathcal{K}_1$ is missing Godbillon-Vey class. The reason is the algebraic nature of the cochains of the latter. We proceed to show that if one allows transcendental cocycles, the Chern class C_1 dissolves by transgression.
Let $\tilde{\mathcal{K}}_1$ be the Hopf algebra obtained by adjoining a primitive element $\log \sigma$ to $\mathcal{K}_1$, subject to the commutation relations

$$[X, \log \sigma] = \sigma^{-1}\sigma^1_{1,1}, \quad [\log \sigma, \sigma_k] = 0, \;\; \forall k \in \mathbb{N}.$$

With the Hopf algebraic structure dictated by the Leibniz rule as follows,

$$\Delta(\log \sigma) = \log \sigma \otimes 1 + 1 \otimes \log \sigma, \tag{6.12}$$

Proposition 6.2. *The 2-cochain*

$$\sqcap_1 := \mathbf{1} \otimes \sigma^{-1} X_1 \otimes \sigma^{-2} \log \sigma \tag{6.13}$$

is a Hochschild cocycle whose Connes boundary is C_1.

Proof. By transfer via the characteristic map χ_τ, we can work in the cyclic complex of $\mathcal{A}_\Gamma$. Denoting the transported cochain by

$$\tilde{\sqcap}_1(f_0 U^*_{\phi_0}, f_1 U^*_{\phi_1}, f_2 U^*_{\phi_2}) = \tau\bigg(f_0 U^*_{\phi_0} \frac{f_1'}{\phi_1'} U^*_{\phi_1} \frac{f_2 \cdot \log(\phi_2')}{\phi_2'} U^*_{\phi_2} \bigg),$$

let us first check that it is a Hochschild cocycle.

$$\begin{aligned}
&b(\tilde{\sqcap}_1)(f_0 U^*_{\phi_0}, \dots, f_3 U^*_{\phi_3}) \\
&= \tau\bigg(f_0 \cdot (f_1 \circ \phi_0) U^*_{\phi_1 \circ \phi_0} \frac{f_2'}{\phi_1'} U^*_{\phi_2} \frac{f_3 \cdot \log(\phi_3')}{\phi_3'} U^*_{\phi_3} \bigg) \\
&- \tau\bigg(f_0 U^*_{\phi_0} \frac{(f_1 \cdot (f_2 \circ \phi_1))'}{(\phi_2 \circ \phi_1)'} U^*_{\phi_2 \circ \phi_1} \frac{f_3 \cdot \log(\phi_3')}{\phi_3'} U^*_{\phi_3} \bigg) \\
&+ \tau\bigg(f_0 U^*_{\phi_0} \frac{f_1'}{\phi_1'} U^*_{\phi_1} \frac{f_2 \cdot (f_3 \circ \phi_2) \cdot \log((\phi_3 \circ \phi_2)')}{(\phi_3 \circ \phi_2)'} U^*_{\phi_3 \circ \phi_2} \bigg) \\
&- \tau\bigg(f_3 \cdot (f_0 \circ \phi_3) U^*_{\phi_0 \circ \phi_3} \frac{f_1'}{\phi_1'} U^*_{\phi_1} \frac{f_2 \cdot \log(\phi_2')}{\phi_2'} U^*_{\phi_2} \bigg).
\end{aligned}$$

By the Leibniz rule we see that,

$$\frac{(f_1 \cdot f_2 \circ \phi_1)'}{(\phi_2 \circ \phi_1)'} = \frac{f_1' \cdot (f_2 \circ \phi_1)}{(\phi_2' \circ \phi_1) \cdot \phi_1'} + \frac{f_1 \cdot (f_2' \circ \phi_1)}{\phi_2' \circ \phi_1}$$

and

$$\frac{f_2 \cdot (f_3 \circ \phi_2) \cdot \log((\phi_3 \circ \phi_2)')}{(\phi_3 \circ \phi_2)'} = \frac{f_2}{\phi_2'} \, \frac{(f_3 \cdot \log(\phi_3')) \circ \phi_2}{\phi_3' \circ \phi_2} + \frac{f_2 \cdot \log(\phi_2')}{\phi_2'} \, \frac{f_3 \circ \phi_2}{\phi_3' \circ \phi_2}$$

which yields that

$$\begin{aligned}
&b(\tilde{\sqcap}_1)(f_0 U^*_{\phi_0}, \dots, f_3 U^*_{\phi_3}) \\
&= \tau\bigg(f_0 U^*_{\phi_0} \frac{f_1'}{\phi_1'} U^*_{\phi_1} \frac{f_2 \cdot \log(\phi_2')}{\phi_2'} U^*_{\phi_2} \frac{f_3}{\phi_3'} U^*_{\phi_3} \bigg) - \tau\bigg(f_3 U^*_{\phi_3} f_0 U^*_{\phi_0} \frac{f_1'}{\phi_1'} U^*_{\phi_1} \frac{f_2 \cdot \log(\phi_2')}{\phi_2'} U^*_{\phi_2} \bigg).
\end{aligned}$$

Finally by the σ^{-1}-tracial property of τ we see $b(\tilde{\sqcap}_1) = 0$.
To show that $B(\tilde{\sqcap}_1) = C_1$, we first observe that $\tilde{\sqcap}_1$ is normalized. So we continue by

$$B(\tilde{\sqcap}_1)(a,b) = \tilde{\sqcap}_1(1,a,b) - \tilde{\sqcap}_1(1,b,a).$$

$$\begin{aligned}&B(\tilde{\sqcap}_1)(f_0U^*_{\phi_0}, f_1U^*_{\phi_1})\\ &= \tau\bigg(\frac{f_0'}{\phi_0'}U^*_{\phi_0}\frac{f_1\cdot\log(\phi_1')}{\phi_1'}U^*_{\phi_1}\bigg) - \tau\bigg(\frac{f_1'}{\phi_1'}U^*_{\phi_1}\frac{f_0\cdot\log(\phi_0')}{\phi_0'}U^*_{\phi_0}\bigg).\end{aligned}$$

Without loss of generality we assume that $\phi_0^{-1} = \phi_1$. Then again by using the σ^{-1}-tracial property of τ we have

$$\begin{aligned}&\tau\bigg(\frac{f_1'}{\phi_1'}U^*_{\phi_1}\frac{f_0\cdot\log(\phi_0')}{\phi_0'}U^*_{\phi_0}\bigg) = \tau\bigg(f_0U^*_{\phi_0}\frac{\log(\phi_0'\circ\phi_1)}{\phi_0'\circ\phi_1}\frac{f_1'}{(\phi_1')^2}U^*_{\phi_1}\bigg)\\ &= -\tau\bigg(f_0U^*_{\phi_0}\frac{f_1'\cdot\log(\phi_1')}{\phi_1'}U^*_{\phi_1}\bigg)\end{aligned}$$

On the other hand one uses the integration by part property of τ to see

$$\begin{aligned}&\tau\bigg(\frac{f_0'}{\phi_0'}U^*_{\phi_0}\frac{f_1\cdot\log(\phi_1')}{\phi_1'}U^*_{\phi_1}\bigg) = -\tau\bigg(f_0U^*_{\phi_0}\frac{1}{\phi_1'}\bigg(\phi_1'\frac{f_1\cdot\log(\phi_1')}{\phi_1'}\bigg)'U^*_{\phi_1}\bigg)\\ &= -\tau\bigg(f_0U^*_{\phi_0}\frac{f_1'\cdot\log(\phi_1')}{\phi_1'}U^*_{\phi_1}\bigg) - \tau\bigg(f_0U^*_{\phi_0}\frac{f_1\cdot\phi_1''}{(\phi_1')^2}U^*_{\phi_1}\bigg).\end{aligned}$$

This completes the proof of the claimed result. ☐

One may also import the Godbillon-Vey cocycle from the Bott bicomplex and show, in a similar fashion as for C_0 and C_1, that it is in the range of the characteristic map

$$\chi_\tau : \ {}^{\sigma^{-1}}\mathbb{C}\otimes\tilde{\mathcal{K}}^{\otimes 2} \to CC^2(\mathcal{A}_\Gamma).$$

Proposition 6.3. *The element*

$$\mathrm{GV} := \mathbf{1}\otimes\log\sigma\otimes\sigma^{-2}\sigma^1_{1,1} - \mathbf{1}\otimes\sigma^{-2}\sigma^1_{1,1}\otimes\sigma^{-1}\log\sigma \in \ {}^{\sigma^{-1}}\mathbb{C}\otimes\tilde{\mathcal{K}}_1\otimes\tilde{\mathcal{K}}_1$$

is a cyclic cocycle.

Proof. On the one hand, one has

$$\begin{aligned}
b(\mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1}) = \mathbf{1} &\otimes 1 \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} \\
&- \mathbf{1} \otimes 1 \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} - \mathbf{1} \otimes \log\sigma \otimes 1 \otimes \sigma^{-2}\sigma^1_{1,1}) \\
&+ \mathbf{1} \otimes \log\sigma \otimes 1 \otimes \sigma^{-2}\sigma^1_{1,1} + \mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} \\
&- \mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} = 0,
\end{aligned} \tag{6.14}$$

and also

$$\begin{aligned}
b(\mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma) = \mathbf{1} &\otimes 1 \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma \\
&- \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} \otimes \sigma^{-1}\log\sigma - \mathbf{1} \otimes 1 \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma \\
&+ \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma \otimes \sigma^{-1} + \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} \otimes \log\sigma \\
&- \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma \otimes \sigma^{-1} \;=\; 0.
\end{aligned} \tag{6.15}$$

On the other hand,

$$\begin{aligned}
\tau_2(\mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1}) &= \mathbf{1} \otimes \Delta S(\log\sigma) \cdot \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} \\
&= -\mathbf{1} \otimes \Delta(\log\sigma) \cdot \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1} \\
&= -(\mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1}\log\sigma \otimes \sigma^{-1} + \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma),
\end{aligned} \tag{6.16}$$

while, taking into account that $\quad S(\sigma^{-2}\sigma^1_{1,1}) = -\sigma^{-1}\sigma^1_{1,1}$,

$$\begin{aligned}
\tau_2(\mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma) &= \mathbf{1} \otimes \Delta S(\sigma^{-2}\sigma^1_{1,1}) \cdot \sigma^{-1}\log\sigma \otimes \sigma^{-1} \\
&= -\mathbf{1} \otimes \Delta(\sigma^{-1}\sigma^1_{1,1}) \cdot \sigma^{-1}\log\sigma \otimes \sigma^{-1} \\
&= -(\mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1}\log\sigma \otimes \sigma^{-1} + \mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1}).
\end{aligned} \tag{6.17}$$

Therefore,

$$\begin{aligned}
&\tau_2(\mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} - \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma) \\
&\qquad = \mathbf{1} \otimes \log\sigma \otimes \sigma^{-2}\sigma^1_{1,1} - \mathbf{1} \otimes \sigma^{-2}\sigma^1_{1,1} \otimes \sigma^{-1}\log\sigma.
\end{aligned} \tag{6.18}$$

$\square$

A complete discussion of the Hopf algebra $\tilde{\mathcal{K}}_n$ and of its Hopf cyclic cohomology will appear in [18].

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